Notes by László Fehér for the lecture on 22 May 2015

1 WZNW Hamiltonian system

Consider a simple Lie group G with Lie algebra $\mathcal{G}$. The WZNW phase space is

$$\mathcal{M} = T^*\widetilde{G} = \widetilde{G} \times \widetilde{\mathcal{G}} = \{\, (g, J) \,|\, g \in \widetilde{G}, \;\; J \in \widetilde{\mathcal{G}} \,\},$$

with loop group

$$\widetilde{G} = C^\infty(S^1, G)$$

and loop algebra

$$\widetilde{\mathcal{G}} = C^\infty(S^1, \mathcal{G}).$$

The symplectic form reads

$$\Omega^\kappa = d\int_0^{2\pi} d\sigma \,\mathrm{Tr}\left(J dg g^{-1}\right) + \frac{\kappa}{2}\int_0^{2\pi} d\sigma\,\mathrm{Tr}\left(dgg^{-1}\right)\wedge\left(dgg^{-1}\right)'$$

Basic objects are the WZNW field g, the 'left-current' J and the 'right-current' I given by

$$I = -g^{-1}Jg + \kappa g^{-1}g'.$$

The currents are the momentum maps that generate two commuting actions of $\widetilde{G}$ on $\mathcal{M}$ that correspond respectively to left and right-translations on $\widetilde{G}$. This means that the following local Poisson bracket relations are valid:

$$\{\,\mathrm{Tr}\,(T_aJ)(\sigma)\,,\mathrm{Tr}\,(T_bJ)(\bar\sigma)\,\} = \mathrm{Tr}\,([T_a,T_b]J)(\sigma)\delta + \kappa\,\mathrm{Tr}\,(T_aT_b)\,\delta'$$

$$\{\,\mathrm{Tr}\,(T_aI)(\sigma)\,,\mathrm{Tr}\,(T_b\,I)(\bar\sigma)\,\} = \mathrm{Tr}\,([T_a,T_b]I)(\sigma)\delta - \kappa\,\mathrm{Tr}\,(T_aT_b)\,\delta'$$

$$\{\,g(\sigma)\,,\mathrm{Tr}\,(T_aJ)(\bar\sigma)\,\} = T_a g(\sigma)\,\delta$$

$$\{\,g(\sigma)\,,\mathrm{Tr}\,(T_aI)(\bar\sigma)\,\} = -g(\sigma)T_a\,\delta,$$

together with $\{J(\sigma), I(\bar\sigma)\} = 0$. Here $\delta = \delta(\sigma - \bar\sigma) = \frac{1}{2\pi}\sum_{n\in\mathbf{Z}} e^{in(\sigma-\bar\sigma)}$ and T_a is a basis of $\mathcal{G}$. The action generated by J is given by

$$L_\gamma : (g, J) \mapsto (\gamma g, \gamma J\gamma^{-1} + \kappa\gamma'\gamma^{-1}), \qquad \gamma\in\widetilde{G}.$$

The action generated by I is written as

$$R_\gamma : (g, J) \mapsto (g\gamma^{-1}, J). \qquad \gamma\in\widetilde{G}.$$

The action L_γ leaves I invariant, while R_γ preserves J and transforms I according to the co-adjoint action (at level $-\kappa$)

$$R_\gamma : I \mapsto \gamma I\gamma^{-1} - \kappa\gamma'\gamma^{-1}.$$

The phase space $\mathcal{M}$ represents the initial data for the WZNW system, whose dynamics is generated by the Hamiltonian

$$H_{\rm WZNW} = \frac{1}{2\kappa}\int_0^{2\pi} d\sigma \, {\rm Tr}\, \left(J^2 + I^2\right).$$

Denoting time by τ and introducing lightcone coordinates as

$$x^\pm := \tau \pm \sigma, \qquad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma),$$

Hamilton's equation can be written in the alternative forms

$$\kappa\partial_+ g = Jg, \quad \partial_- J = 0 \qquad \Leftrightarrow \qquad \kappa\partial_- g = -gI, \quad \partial_+ I = 0.$$

These equations imply

$$\partial_-(\partial_+ g \cdot g^{-1}),$$

which is easily solved as follows:

$$g(\tau,\sigma) = g_L(x^+) g_R(x^-)$$

where $g_L(x^+ + 2\pi) = g_L(x^+)\eta$ and $g_R(x^- + 2\pi) = \eta g_R(x^-)$ with some 'monodromy matrix' $\eta \in G$.

Remark 1. The problem of chiral separation is to find good Poisson structure on $\mathcal{M}_L = \{g_L\}$ and $\mathcal{M}_R = \{g_R\}$ such that $\mathcal{M}$ is recovered from $\mathcal{M}_L \times \mathcal{M}_R$ upon 'imposing monodromy constraint' $\eta_L = \eta_R$ (ensuring the 2π-periodicity $g(\tau, \sigma + 2\pi) = g(\tau,\sigma)$).

Remark 2. The Poisson bracket version of the Virasoro algebra is

$$\{W_1(\sigma), W_1(\bar\sigma)\} = -W_1'(\bar\sigma)\delta + 2W_1(\bar\sigma)\delta' + C\delta'''$$

where C is some constant. A periodic (real or complex) 'phase-space function' $W_a(\sigma)$ is called a primary field with weight Δ_a if

$$\{W_1(\sigma), W_a(\bar\sigma)\} = -W_a'(\bar\sigma)\delta + \Delta_a W_a(\bar\sigma)\delta'.$$

A classical $\mathcal{W}$-algebra is a Poisson algebra generated by a finite number of fields $W_1, W_2, \dots, W_N$ such that the above relations hold as well as

$$\{W_b(\sigma), W_c(\bar\sigma)\} = \sum_i P^i_{bc}(W_1(\bar\sigma), \dots, W_N(\bar\sigma))\delta^{(i)}(\sigma - \bar\sigma).$$

such that on the right hand side we have a finite sum and the P^i_{bc} are differential polynomials in the basic fields (with constant terms allowed).

2 The WZNW $\longrightarrow$ Toda reduction

Let $\mathcal{G}$ now be a spit real form of a complex simple Lie algebra (or we can tolerate to work with the complex field) and G a corresponding connected Lie group. Let S_0 be a diagonalizable element of $\mathcal{G}$ that is part of an sl_2 triple

$$\mathcal{S} = \{S_-, S_0, S_+\} \subset \mathcal{G} \quad \text{with} \quad [S_0, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 2S_0.$$

Suppose (for simplicity) that ad_{S_0} has only integer eigenvalues in $\mathcal{G}$. Consider the triangular decomposition

$$\mathcal{G} = \mathcal{G}_- + \mathcal{G}_0 + \mathcal{G}_+$$

induced by the sign of the eigenvalues of ad_{S_0}, which also gives an integral grading

$$\mathcal{G} = \oplus_m \mathcal{G}_m.$$

Let $G_0, G_\pm$ denote the connected subgroups of G corresponding to the subalgebras $\mathcal{G}_0, \mathcal{G}_\pm$. Then there is an open subset $\check{G} \subset G$ whose elements can be uniquely written in the form

$$g = g_+ g_0 g_- \quad \text{with} \quad g_{0,\pm} \in G_{0,\pm}.$$

This is called 'generalized Gauss decomposition'.

Now define the (conformal) Toda field equation associated with the sl_2-triple to be the following equation for a G_0-valued field $g_0(\tau, \sigma)$:

$$\partial_-(\partial_+ g_0 \cdot g_0^{-1}) = [S_-, g_0 S_+ g_0^{-1}].$$

Claim. The above Toda equation is just the WZNW field equation for a 'Gauss decomposable' G-valued field subject to the following constraints:

$$\pi_-(J) = S_- \quad \text{and} \quad \pi_+(I) = -S_+,$$

where $\pi_{\pm,0} : \mathcal{G} \to \mathcal{G}_{\pm,0}$ denote the projection operators corresponding to the triangular decomposition of $\mathcal{G}$. (Here we put $\kappa = 1$.)

I explain this reduction at the level of the equation of motion, then indicate that it is actually an example of 'phase space reduction by first class constraints'. From the point of view of the phase space, it is simply a case of Marsden-Weinstein reduction, but the WZNW Hamiltonian is 'only' weakly invariant (which is sufficient). I also detail the 'principal special case'.

Remark. All this works also if one replaces S_0 by an arbitrary integral grading operator H and replaces $S_\pm$ by arbitrary elements of H-grade ± 1, but the systems associated with sl_2-triples represent the nicest class.

3 'Classical Wakimoto realizations' from reduction

Let $\mathcal{G}$ be a complex (or real split) simple Lie algebra and $\mathcal{G} = \oplus_m \mathcal{G}_m$ an integral gradation, with associated 'triangular' decomposition

$$\mathcal{G} = \mathcal{G}_- + \mathcal{G}_0 + \mathcal{G}_+.$$

Similarly as before, denote by $G_0, G_\pm$ the corresponding connected subgroups of a connected Lie group G with Lie algebra $\mathcal{G}$. Moreover, introduce the parabolic subgroup $P < G$ and its Lie algebra:

$$P = G_0 G_+, \qquad \mathcal{P} = (\mathcal{G}_0 + \mathcal{G}_+).$$

We identify

$$\mathcal{G}^* \simeq \mathcal{G}, \qquad \mathcal{G}_-^* \simeq \mathcal{G}_+, \qquad \mathcal{G}_0^* \simeq \mathcal{G}_0$$

by means of the (normalized) Killing form. By using right-trivialization, we have

$$T^*G_- = G_- \times \mathcal{G}_-^* \simeq G_- \times \mathcal{G}_+ = \{(g_-, j)\}.$$

3.1 The finite-dimensional case

Claim. The map

$$I : T^*G_- \times \mathcal{G}_0 \to \mathcal{G}$$

given by the formula

$$I(g_-, j, j_0) = g_-^{-1}(-j + j_0)g_-$$

is a Poisson map.

Proof: 'Clever' application of symplectic reduction of $T^*\check{G}$. The symmetry group used in the reduction is P. Note that notationwise we 'pretend' to deal with a matrix Lie group. It is instructive to describe the result in terms of Darboux coordinates on T^*G_- as well.

3.2 The infinite-dimensional case

Let j_0 be an affine Kac-Moody current at level $-\kappa$, which means that it obeys the Poisson bracket relations

$$\{\,\mathrm{Tr}\,(Y_i\, j_0(\sigma))\,,\, \mathrm{Tr}\,(Y_l\, j_0(\bar\sigma))\,\} = \mathrm{Tr}\,([Y_i, Y_l]\, j_0(\sigma))\delta - \kappa\,\mathrm{Tr}\,(Y_i\, Y_l)\,\delta',$$

where Y_i is a basis of $\mathcal{G}_0$. Consider also the cotangent bundle

$$T^*\widetilde{G}_- = \widetilde{G}_- \times \widetilde{\mathcal{G}}_+ = \{(g_-, j) \mid g_- \in C^\infty(S^1, G_-),\ j \in C^\infty(S^1, \mathcal{G}_+)\,\},$$

on which the non-vanishing fundamental Poisson bracket read as follows:

$$\begin{aligned}\{\,\mathrm{Tr}\,(V_\alpha\, j)(\sigma)\,,\ \mathrm{Tr}\,(V_\beta\, j)(\bar\sigma)\,\} &= \mathrm{Tr}\,([V_\alpha, V_\beta]\, j)(\sigma)\,\delta,\\ \{\,g_-(\sigma)\,,\ \mathrm{Tr}\,(V_\alpha\, j)(\bar\sigma)\,\} &= V_\alpha\, g_-(\sigma)\,\delta,\end{aligned}$$

where V_α is a basis of $\mathcal{G}_-$. Denote by V^β the dual basis of $\mathcal{G}_+$, $\mathrm{Tr}\,(V^\beta V_\alpha) = \delta^\beta_\alpha$. Let q_α be some global coordinates on G_-. Define

$$N^{\alpha\beta}(q) \equiv \mathrm{Tr}\,(V^\beta \frac{\partial g_-}{\partial q_\alpha} g_-^{-1}).$$

Then

$$j = j(q,p) = N^{-1}_{\alpha\beta}(q) p^\beta V^\alpha$$

in terms of the canonical (Darboux) coordinates $q_\alpha(\sigma)$, $p^\beta(\sigma)$ on $T^*\widetilde{G}_-$. Note that

$$\int_{S^1} d\,\mathrm{Tr}\left(j dg_- g_-^{-1}\right) = \int_{S^1} d\left(p^\alpha dq_\alpha\right)\,,$$

and therefore

$$\{q_\alpha(\sigma), p^\beta(\bar\sigma)\} = \delta^\beta_\alpha \delta(\sigma - \bar\sigma).$$

The classical Wakimoto realization is given by the statement:

Claim. *As a consequence of the PBs of j_0 and q_α, p^β above, the 'classical Wakimoto current'*

$$I(q,p,j_0) = I(g_-(q), j(q,p), j_0) = g_-^{-1}(-j + j_0)g_- + \kappa g_-^{-1} g_-'$$

satisfies the affine Kac-Moody Poisson brackets

$$\{\,\mathrm{Tr}\,(T_a\, I)(\sigma)\,,\ \mathrm{Tr}\,(T_b\, I)(\bar\sigma)\,\} = \mathrm{Tr}\,([T_a, T_b]\, I)(\sigma)\delta - \kappa\,\mathrm{Tr}\,(T_a\, T_b)\,\delta',$$

where $\{T_a\}$ is a basis of $\mathcal{G}$. The affine 'Sugawara-density' is quadratic in the 'free field constituents',

$$\frac{1}{2\kappa}\mathrm{Tr}\,(I^2) = \frac{1}{2\kappa}\mathrm{Tr}\,(j_0^2) - \mathrm{Tr}\,(j g_-' g_-^{-1}) = \frac{1}{2\kappa}\mathrm{Tr}\,(j_0^2) - \sum_\alpha p^\alpha q_\alpha'.$$

Proof: By reduction of the (big cell of the) WZNW phase space $T^*\widetilde{G}$ definded by retricting the G-valued WZNW field to vary in $\check{G} = G_+ G_0 G_-$. The symmetry group used in the reduction is $\widetilde{P}$.

Remark. The generalized Wakimoto realization of the affine Kac-Moody algebra used in CFT is obtained by direct quantization. (Due to normal ordering ambiguities, there is a quantum correction, which – with Jan de Boer – we have determined explicitly. The procedure requires choosing q_α to be 'upper triangular coordinates' on G_-. For example, exponential coordinates associated with any graded basis of $\mathcal{G}_-$ are appropriate.

4 Classical $\mathcal{W}$-algebras from reduction

We again use the sl_2-triple S_-, S_0, S_+ and for simplicity assume that ad_{S_0} has only integer eigenvalues.

We consider the affine Kac-Moody current algebra

$$\{\,\mathrm{Tr}\,(T_a J)(\sigma)\,,\mathrm{Tr}\,(T_b J)(\bar{\sigma})\,\} = \mathrm{Tr}\,([T_a,T_b]J)(\sigma)\delta + \kappa\,\mathrm{Tr}\,(T_a T_b)\,\delta'$$

on the phase space $C^\infty(S^1,\mathcal{G})$. Now we apply reduction by using the Hamiltonian action of the group

$$N = C^\infty(S^1, G_+),$$

which operates by the formula:

$$L_\gamma : J \mapsto \gamma J \gamma^{-1} + \kappa\gamma'\gamma^{-1}, \qquad \gamma\in N.$$

This is generated by the momentum map

$$J \mapsto \pi_-(J).$$

The reduction of interest is defined by imposing the (first class) momentum map constraint[1]

$$\pi_-(J) = S_-.$$

The (isotropy) gauge group is the full group $C^\infty(S^1,G_+)$, and therefore the reduced phase space is

$$\{J = S_- + j \mid j\in C^\infty(S^1,\mathcal{G}_0+\mathcal{G}_+\}/N.$$

Basic algebraic fact:

$$\mathcal{G}_0\oplus\mathcal{G}_+ = [\mathcal{G}_+, S_-]\oplus \mathrm{Ker}(\mathrm{ad}_{\mathrm{S}_+}).$$

Claim 1. The action of N on the 'constraint surface' is free. The subspace of constrained currents given by

$$\mathcal{W} = \{J = S_- + W \mid W\in C^\infty(S^1,\mathrm{Ker}(\mathrm{ad}_{\mathrm{S}_+}))\}$$

is a global cross section for the N-action, and thus yields a model of the reduced phase space. For any $J=(S_-+j)$, the equation

$$\gamma(S_-+j)\gamma^{-1} + \kappa\gamma'\gamma^{-1} = S_- + W, \quad W\in C^\infty(S^1,\mathrm{Ker}(\mathrm{ad}_{\mathrm{S}_+}))$$

has a unique solution for γ. The components of the resulting $\log\gamma$ are differential polynomials in the components of j. Consequently, the ring of gauge invariant differential polynomials of j is freely generated by the components of resulting function $W = W(j)$.

[1]This is the same that was used in the WZNW⟶Toda reduction!

Claim 2. Take a homogeneous basis $\{F_a\}$ of $\mathrm{Ker}(\mathrm{ad}_{S_+})$,

$$F_1 := \frac{\kappa}{\mathrm{Tr}\,(S_+S_-)} S_+, \quad [S_0, F_a] := (\Delta_a - 1)F_a,$$

and decompose

$$W(j) = \sum_a W_a(j) F_a.$$

Then the induced Poisson brackets of the gauge invariant differential polynomials

$$W_a(\sigma) = W_a(j(\sigma))$$

yield a classical $\mathcal{W}$-algebra, where W_1 is a Virasoro density and W_a for $a \neq 1$ is a primary field of conformal weight Δ_a.

Remark 1. The Virasoro density W_1 is inherited from the modified Sugawara density,

$$L = \frac{1}{2\kappa} \mathrm{Tr}\,(J^2) - \mathrm{Tr}\,(S_0 J')$$

which yields a gauge invariant differential polynomial on the constraint surface. The global cross section exhibited in 'Claim 1' is often called the 'highest weight gauge'.

Remark 2. It follows that the conformal Toda field theories realize the chiral $\mathcal{W}$-algebras as their symmetry structures.

Remark 3. The case of half-integral sl_2-triples is a small variation, which I can explain at the blackboard.

5 WZNW factory of classical dynamical r-matrices

The main products of this factory are the possible Poisson structures carried by the chiral WZNW fields g_L and g_R, regarded as independent, i.e., 'before' imposing the monodromy constraint. These Poisson structures are encoded by interesting dynamical r-matrices, which solve various generalizations of the classical Yang-Baxter equation. See e.g. my review in arXiv:hep-th/0212006.

6 Some references

My lecture was based mainly on the papers listed below. Unfortunately, the conventions of these papers are not always the same. Here, I tried to follow a unified set of conventions.

For the WZNW Hamiltonian system, see e.g. hep-th/9912173 and references therein.

Section 2 was based on L. Feher et al: Ann. Phys. 213, 1-20 (1992)). There are still many open problems about global issues, which are studied in the lowest rank case in the paper arXiv:hep-th/9703045.

Our Wakimoto stuff is contained in arXiv:hep-th/9605102, arXiv:hep-th/9611083 and in arXiv:math/0305268 (the last paper is phrased in the language of vertex algebras).

About classical $\mathcal{W}$-algebras, the 'highest weight gauge' was originally found in the case of the principal sl_2 subalgebra in J. Balog et al: Ann. Phys. 203, 76-136 (1990). The case of a general sl_2 embeddings is treated, for example, in arXiv:hep-th/9304125.

Final remark. If anybody wishes it, I am happy to explain the proof of any of the 'claims' and remarks formulated in these notes. Actually I have prepared to give proofs on Friday, but badly miscalculated the time. I should also note that not all 'claims' were our results. Proper credits are given in the references. (If somebody is interested, I can send the references that are not in the arXiv.)