Euler characteristics of Hilbert schemes of points on simple singularities

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References

1. arXiv:1510.02677
2. arXiv:1512.06844
3. arXiv:1512.06848

Some parts are joint works with my supervisors, András Némethi and Balázs Szendrői.

Outline

Hilbert scheme of points

Let X be a quasiprojective variety over $\mathbb{C}$.

Definition (Theorem)

For every $n \in \mathbb{N}$ there is a Hilbert scheme $\mathrm{Hilb}^n(X)$, which parametrizes 0 dimensional subschemes (ideal sheaves) of colength n on X.

Remark

1. $\mathrm{Hilb}^n(X)$ *represents a moduli functor.*
2. *Every* $Z \in \mathrm{Hilb}^n(X)$ *decomposes as* $Z = \coprod Z_j$, *where the supports* $P_j = \mathrm{Supp} Z_j$ *are mutually disjoint.*
3. $\mathrm{colength}(Z, P_j) = \mathrm{length}(\mathcal{O}_{Z,P_j})$.
4. *Hilbert-Chow morphism*

$$\Pi : \mathrm{Hilb}^n(X) \to S^n X, \quad I \mapsto \sum_j \mathrm{colength}(Z, P_j) P_j$$

Hilbert scheme of points

Relative version: For $Y \subset X$ a (locally) closed subvariety: $\mathrm{Hilb}^n(X,Y) \subset \mathrm{Hilb}^n(X)$ is the Hilbert scheme of points (set-theoretically) supported on Y.

Question: From topological/analytical properties of X what can we infere about the topology of $\mathrm{Hilb}^n(X)$?

Usually: better to work with the collection of the Hilbert schemes for all n together.

Curves

C curve over $\mathbb{C}$ with singularities $p_i \rightsquigarrow C_{sm} = C \setminus \coprod_i p_i$ smooth part

$$
\begin{aligned}
Z_C &= \sum_{n=0}^{\infty} \chi(\mathrm{Hilb}^n(C)) q^n \\
&= \sum_{n=0}^{\infty} \sum_{n_0+\cdots=n} \chi(\mathrm{Hilb}^{n_0}(C_{sm})) q^{n_0} \prod_i \chi(\mathrm{Hilb}^{n_i}(C, p_i)) q^{n_i} \\
&= Z_{C_{sm}}(q) \prod_i \underbrace{\left(\sum_{n=0}^{\infty} \chi(\mathrm{Hilb}^n(C, p_i)) q^n \right)}_{Z_{(C,p_i)}(q)}
\end{aligned}
$$

Theorem (Macdonald)

$$
Z_{C_{sm}}(q) = \frac{1}{(1-q)^{\chi(C_{sm})}}
$$

Plane curve singularities

$(C, 0)$ a plane curve singularity with link $L_{(C,0)} \subseteq S^3$ and Milnor-number μ

Theorem (Maulik, conjecture of Oblomkov-Shende)

$$Z_{(C,0)}(q^2) = \frac{1}{q^2}\left[\left(\frac{q}{a}\right)^{\mu} P(L_{(C,0)})\right]\Big|_{a=0},$$

where $P(L) \in \mathbb{Z}[a^{\pm}, (q - q^{-1})^{\pm}]$ *is the HOMFLY polynomial of the link L.*

Corollary

The topology of the link (i.e. its embedding type) determines $Z_{(C,p)}(q)$

Smooth surfaces

Theorem (Fogarty)

If X is a smooth surface over $\mathbb{C}$ then:

1. $\mathrm{Hilb}^n(X)$ *is smooth of dimension* $2n$
2. $\Pi : \mathrm{Hilb}^n(X) \to S^n X$ *is a resolution of singularities*

Theorem (Göttsche)

$$\sum_{n=0}^{\infty} P_t(\mathrm{Hilb}^n(X)) q^n$$

$$= \prod_{n=1}^{\infty} \frac{(1+t^{2n-1}q^n)^{b_1(X)}(1+t^{2n+1}q^n)^{b_3(X)}}{(1+t^{2n-2}q^n)^{b_0(X)}(1+t^{2n}q^n)^{b_2(X)}(1+t^{2n+2}q^n)^{b_4(X)}}$$

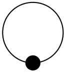
Corollary

$$Z_X(q) = \prod_{i=1}^{\infty} \frac{1}{(1-q^n)^{\chi(S)}}$$

Affine plane

Theorem (Barth, Nakajima)

$\mathrm{Hilb}^n(\mathbb{C}^2)$ *is the quiver variety corresponding to the Jordan quiver*



with dimension vectors $\mathbf{v} = (n)$, $\mathbf{w} = (1)$.

That is

$$\mathrm{Hilb}^{\mathrm{n}}(\mathbb{C}^2) = \{(B_1, B_2, i, j) \mid [B_1, B_2] + ij = 0\} // \mathrm{Gl}_n(\mathbb{C}) ,$$

where $B_1, B_2 \in \mathrm{End}(\mathbb{C}^n)$, $i \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n)$, $j \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$, $g \in \mathrm{Gl}_n(\mathbb{C})$ *acts as*

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}) ,$$

and // is the GIT quotient for some stability condition.

Affine plane

$$Z_{\mathbb{C}^2}(q) = \prod_{i=1}^{\infty} \frac{1}{(1-q^n)} = \sum_{n\geq 0} p(n) q^n \ .$$

This is the character formula for $\mathcal{F}$, the Fock space representation of the Heisenberg algebra.
Recall:

- $\mathcal{F} = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{C}\lambda$,
- $p(\lambda) = \sum_{\text{1 block added}} \lambda'$,
- $q(\lambda) = \sum_{\text{1 block removed}} \lambda'$,
- $[p, q] = Id$.

Theorem (Nakajima, Grojnowski)

$$H^*(\mathrm{Hilb}(\mathbb{C}^2)) = \bigoplus_{n=0}^{\infty} H^*(\mathrm{Hilb}^n(\mathbb{C}^2)) \cong \mathcal{F} \ .$$

Coarse Hilbert scheme

$G \subset \mathrm{Sl}_2(\mathbb{C})$ finite subgroup, acting on $\mathbb{C}^2$.
$\mathbb{C}^2/G$ qoutient variety with an orbifold structure.

Definition

Coarse (invariant) Hilbert scheme:

$$\mathrm{Hilb}(\mathbb{C}^2/G) = \{Z \lhd \mathbb{C}[x,y]^G | Z \text{ is of finite colength}\} .$$

As before, this decomposes as

$$\mathrm{Hilb}(\mathbb{C}^2/G) = \coprod_{m \in \mathbb{N}} \mathrm{Hilb}^m(\mathbb{C}^2/G) .$$

Orbifold Hilbert schemes

Definition

Orbifold (equivariant) Hilbert scheme:

$$\mathrm{Hilb}([\mathbb{C}^2/G]) = \{I \in \mathrm{Hilb}(\mathbb{C}^2) \mid I \text{ is } G\text{-invariant}\}\ .$$

This stratifies as

$$\mathrm{Hilb}([\mathbb{C}^2/G]) = \bigcup_{\rho \in \mathrm{Rep}(G)} \mathrm{Hilb}^{\rho}([\mathbb{C}^2/G])\ ,$$

where

$$\mathrm{Hilb}^{\rho}([\mathbb{C}^2/G]) = \{I \in \mathrm{Hilb}(\mathbb{C}^2) \mid H^0(\mathcal{O}_I) = H^0(\mathcal{O}_{\mathbb{C}^2}/I) \simeq_G \rho\}\ .$$

Generating series

Let $\rho_0, \ldots \rho_n$ be the irreducible representations of G.

Definition

(a) Coarse generating series (or coarse partition function):

$$Z_{\mathbb{C}^2/G}(q) = \sum_{m=0}^{\infty} \chi\left(\mathrm{Hilb}^m(\mathbb{C}^2/G)\right) q^m .$$

(b) Orbifold generating series (or Orbifold partition function):

$$Z_{[\mathbb{C}^2/G]}(q_0, \ldots, q_n) =$$

$$= \sum_{m_0,\ldots,m_n=0}^{\infty} \chi\left(\mathrm{Hilb}^{m_0\rho_0+\ldots+m_n\rho_n}([\mathbb{C}^2/G])\right) q_0^{m_0} \cdot \ldots \cdot q_n^{m_n} .$$

Maps between the orbifold and coarse Hilbert schemes

$i : \mathbb{C}[x,y]^G \subset \mathbb{C}[x,y]$ inclusion.

Definition

(a) Pushforward (scheme-theoretic):

$$p_* : \mathrm{Hilb}([\mathbb{C}^2/G]) \to \mathrm{Hilb}(\mathbb{C}^2/G), \quad J \mapsto J^G = J \cap \mathbb{C}[x,y]^G .$$

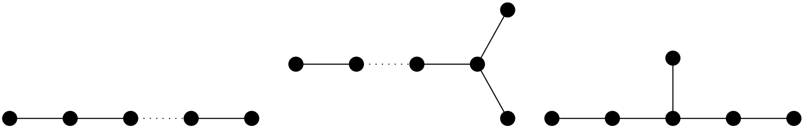
(b) Pullback (only set-theoretic):

$$i^* : \mathrm{Hilb}(\mathbb{C}^2/G)(\mathbb{C}) \to \mathrm{Hilb}([\mathbb{C}^2/G])(\mathbb{C}), \quad I \mapsto i^*I = \mathbb{C}[x,y].I.$$

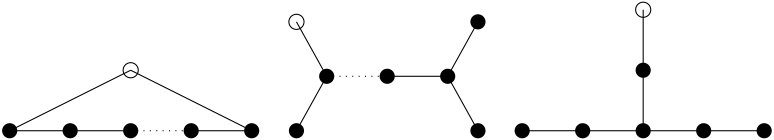
$(\mathbb{C}[x,y].I)^G = I$ for any $I \lhd \mathbb{C}[x,y]^G \Longrightarrow p_* \circ i^*$ is the identity.

McKay correspondence

Finite subgroups of $\mathrm{Sl}_2(\mathbb{C})$: $A_n (n \geq 1)$, $D_n (n \geq 4)$, E_6, E_7, E_8.
To the quotient $\mathbb{C}^2/G$ we can associate its resolution graph:



Denote the corresponding simple (finite dimensional) Lie algebra by $\mathfrak{g}$, the normalized Killing form on it by $(\ |\ )$.
The (irred.) representations of the finite group are described by another graph (McKay quiver, $\rho_{\mathrm{def}} \otimes \rho_i = \bigoplus_j a_{ij} \rho_j$):



Affine Lie algebras

$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$

- $[X \otimes z^m, Y \otimes z^n] = [X, Y] \otimes z^{m+n} + m\delta_{m+n,0}(X|Y)c$
- c is central
- $d = -z\frac{d}{dz}$

Example

$\hat{\mathfrak{u}}(1) = \mathfrak{heis}$ the Heisenberg algebra.

Affine simple roots: $\hat{\Delta} = \{\alpha_0, \alpha_1, \ldots, \alpha_n\} = \{\alpha_0\} \cup \Delta$.
There is a natural scalar product on these $\rightsquigarrow$ identification with the dual space.
Fundamental weights: $\{\omega_0, \omega_1, \ldots, \omega_n\}$.
Level l representation: when c acts as multiplication by l.

Affine Lie algebra action on the homologies

Let V_0 be the level-1 representation with highest weight ω_0 (*basic representation*).
Let $\mathcal{F}$ be the standard Fock space representation of $\mathfrak{heis}$.
Then $V = V_0 \otimes \mathcal{F}$ is a representation of $\hat{\mathfrak{g}} \oplus_c \mathfrak{heis}$ (*extended basic representation*), where $\oplus_c = \oplus +$ centers identified.

Theorem (Nakajima)

$H^*(\mathrm{Hilb}([\mathbb{C}^2/G])$ carries an action of $\hat{\mathfrak{g}} \oplus_c \mathfrak{heis}$, under which it is graded isomorphic to V.

Decomposition of V

$Q = \mathbb{Z}\Delta$ root lattice, P weight lattice.

Theorem (Frenkel-Kac)

$$V \cong \mathcal{F}^{n+1} \otimes \mathbb{C}[Q]\,.$$

Corollary (Weyl-Kac)

$$\mathrm{char}_V(q_0, \dots, q_n) = \sum_{\lambda \ occours \ in \ V} \mathrm{mult}(\lambda)\mathrm{e}^{\lambda} =$$

$$\frac{\sum_{\overline{m}=(m_1,\dots,m_n)\in\mathbb{Z}^n}^{\infty} q_1^{m_1} \dots q_n^{m_n} (q^{1/2})^{\overline{m}^\top \cdot C \cdot \overline{m}}}{\prod_{m=1}^{\infty}(1-q^m)^{n+1}}\,,$$

where $\mathrm{e}^{\alpha_i} = q_i$, $\mathrm{e}^{-\delta} = \mathrm{e}^{\sum_i a_i \alpha_i} = q$, C *is the finite Cartan matrix.*

Since there are no odd cohomologies, we have

$$Z_{[\mathbb{C}^2/G]}(q_0, \dots, q_n) = char_V(q_0, \dots, q_n)\,.$$

Affine crystals

Recall: the partitions (Young diagrams) give a basis for the Fock-space.

Definition (Theorem)

1. When $\mathfrak{g}$ is of type A_n or D_n, then V can be constructed on a vector space, which is spanned by a *crystal basis*.
2. The elements of the crystal basis are in one-to-one correspondence with a set $\mathcal{Z}$ of combinatorial objects, called *Young walls*.

Cell decomposition for orbifold Hilbert schemes

Theorem (Gy-N-Sz)

Let $[\mathbb{C}^2/G]$ be a simple singularity orbifold, where G is of type A_n for $n \geq 1$ or D_n for $n \geq 4$. Then there is an explicit decomposition of $\mathrm{Hilb}([\mathbb{C}^2/G])$ into affine cells indexed by the set of Young walls $\mathcal{Z}$ of the appropriate type.

Corollary

$$Z_{[\mathbb{C}^2/G]}(q_0, \dots, q_n) = \sum_{\lambda \in \mathcal{Z}} \prod_{j=0}^{n} q_j^{w_j(\lambda)}$$

Remark

- *For A_n this was done already by Fujii-Minabe.*
- *This strengthens Nakajima's result.*
- *The RHS of the prevoius character formula enumerates the Young walls of the appropriate type.*

Cell decomposition for coarse Hilbert schemes

Theorem (Gy-N-Sz)

Let $\mathbb{C}^2/G$ be a simple singularity orbifold, where G is of type A_n for $n \geq 1$ or D_n for $n \geq 4$. Then there is specific, combinatorially defined subset $\mathcal{Z}^0 \subset \mathcal{Z}$, and an explicit decomposition of $\mathrm{Hilb}(\mathbb{C}^2/G)$ into affine cells indexed by the set of Young walls $\mathcal{Z}^0$ of the appropriate type. Moreover, there is a combinatorially defined mapping $\mathcal{Z} \to \mathcal{Z}^0$, such that the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Hilb}([\mathbb{C}^2/G]) & \longrightarrow & \mathcal{Z} \\ \big\downarrow p_* & & \big\downarrow p \\ \mathrm{Hilb}(\mathbb{C}^2/G) & \longrightarrow & \mathcal{Z}^0 \end{array},$$

where the horizontal maps associate to an ideal the Young wall of its cell.

The orbifold A_n case

G = cyclic subgroup of $\mathrm{Sl}_2(\mathbb{C})$ of order $n+1$.
Generated by

$$\sigma = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix},$$

where ω is a $(n+1)$-st root of unity.
All irreducible representations of G are one dimensional.
They are given by $\rho_j : \sigma \mapsto \omega^j$, for $j \in \{0, \ldots, n\}$.
σ commutes with the diagonal two torus $T = (\mathbb{C}^2)$
$\implies T \curvearrowright [\mathbb{C}^2/G]$, $T \curvearrowright \mathbb{C}^2/G$
$\implies T \curvearrowright \mathrm{Hilb}([\mathbb{C}^2/G])$, $T \curvearrowright \mathrm{Hilb}(\mathbb{C}^2/G)$.

$(\mathbb{C}^*)^2$ fixpoints

The Young wall pattern of type A_n:

$\vdots$							
0	1						
1	2						
$\vdots$	$\vdots$						
n	0		$n-2$	$n-1$	n	0	
0	1	$\cdots$	$n-1$	n	0	1	$\cdots$

$\mathcal{Z}$ = Young diagrams with this coloring.

For $\lambda \in Z$, let $w_j(\lambda)$ denote the number of blocks in λ labeled j.

Multi-weight: $w(\lambda) = (w_0(\lambda), \ldots, w_n(\lambda))$.

Proposition

- *Affine cells of* $\mathrm{Hilb}([\mathbb{C}^2/G]) \leftrightarrow \mathrm{Hilb}([\mathbb{C}^2/G])^T \leftrightarrow \mathcal{Z}$.
- $H^0(O_I) = \bigoplus_i \rho_i^{\oplus m_i}$ *at an ideal* I *which is a fixpoint described by* λ *if and only if* $(m_0, \ldots, m_n) = w(\lambda)$.

Idea of proof

- $\mathbb{C}[x,y]^T$ = monomials.
- $\mathrm{Hilb}([\mathbb{C}^2/G])^T = \mathrm{Hilb}(\mathbb{C}^2)^T$ = monomial ideals.
- Choose a generic 1D-subtorus $T_0 \subset T$ which has positive weight on x and y.
- Bialynizcki-Birula → all limits of T_0-orbits at $t = 0$ exist (eventhough $\mathrm{Hilb}([\mathbb{C}^2/G])$ is not compact).
- Take the BB decomposition of $\mathrm{Hilb}([\mathbb{C}^2/G])$.

Corollary

Cells of $\mathrm{Hilb}(\mathbb{C}^2/G) \leftrightarrow \mathrm{Hilb}(\mathbb{C}^2/G)^T \leftrightarrow$ *monomial ideals in* $\mathbb{C}[x,y]^G = \mathbb{C}[x^{n+1}, xy, y^{n+1}] \leftrightarrow$ *0-generated Young walls (where all the generators are of color 0)* $=: \mathcal{Z}^0$.

The A_n-abacus

The *abacus of type A_n* is:

$$\begin{array}{ccccc} \vdots & \vdots & & \vdots & \vdots \\ -n & -n+1 & \dots & -1 & 0 \\ 1 & 2 & \dots & n & n+1 \\ \vdots & \vdots & & \vdots & \vdots \end{array}$$

To a diagram $\lambda \in \mathcal{Z}$ associate the infinite series of numbers: $\{\lambda_1, \lambda_2 - 1, \lambda_3 - 2, \dots\}$
We put a bead on each of these numbers.

Example (n=2)

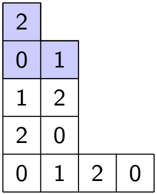
2			
0	1		
1	2		
2	0		
0	1	2	0

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \textcircled{-8} & \textcircled{-7} & \textcircled{-6} \\ \textcircled{-5} & \textcircled{-4} & -3 \\ \textcircled{-2} & \textcircled{-1} & 0 \\ 1 & 2 & \textcircled{3} \\ 4 & \textcircled{5} & 6 \\ \vdots & \vdots & \vdots \end{array}$$

Cores

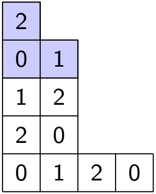
Border strip: a skew Young diagram which does not contain 2×2 blocks and which contains exactly one j-labelled block for all labels j.

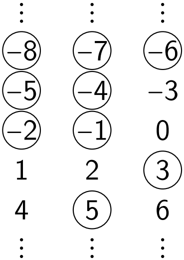
Example



Observation: Removal of a border strip corresponds to shifting a bead up in the abacus representation.
Aim: we remove the border strips until possible $\rightsquigarrow$ core.

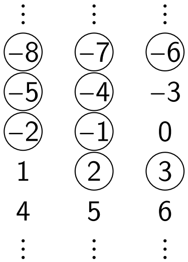
Example





Example

1	2		
2	0		
0	1	2	0



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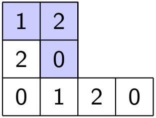

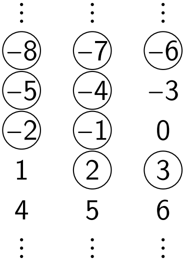

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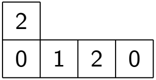

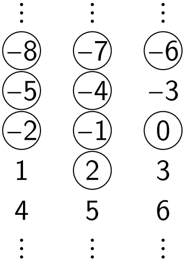

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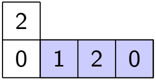
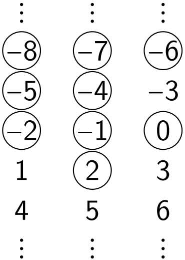




Example

2
0

⋮	⋮	⋮
−8	−7	−6
−5	−4	−3
−2	−1	0
1	2	3
4	5	6
⋮	⋮	⋮

Structure of the orbifold generating series

The relative positions to the highest possible places of the beads on a single ruler are described by a partition: $\mu_1 \geq \cdots \geq \mu_r \geq 0$.
There are $n+1$ rulers $\Longrightarrow$ for each core possible extensions by $n+1$ partitions (*quotients*).
A border strip has weight $q = q_0 \dots q_n$.

Corollary

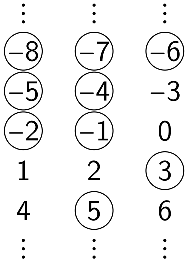
- *$\mathcal{Z} \longleftrightarrow \mathcal{C} \times \mathcal{P}^{n+1}$ where $\mathcal{C}$ is the set of cores (diagrams without border strips).*
- $$Z(q_0, \dots, q_n) = Z_{\mathrm{quotients}}(q) Z_{\mathrm{cores}}(q_0, \dots, q_n) =$$
  $$= \left(\prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{n+1} \sum_{\overline{m}=(m_1,\dots,m_n)\in\mathbb{Z}^n}^{\infty} q_1^{m_1} \dots q_n^{m_n} (q^{1/2})^{\overline{m}^{\top} \cdot C \cdot \overline{m}},$$
  where C is the Cartan matrix of type A.

Combinatorial description of the pushforward map

The pushforward map was induced by $I \mapsto I^G = I \cap \mathbb{C}[x,y]^G$. Thus, we have to extend the diagram of I to get the smallest 0-generated diagram which covers it.

Example

2			
0	1		
1	2		
2	0		
0	1	2	0



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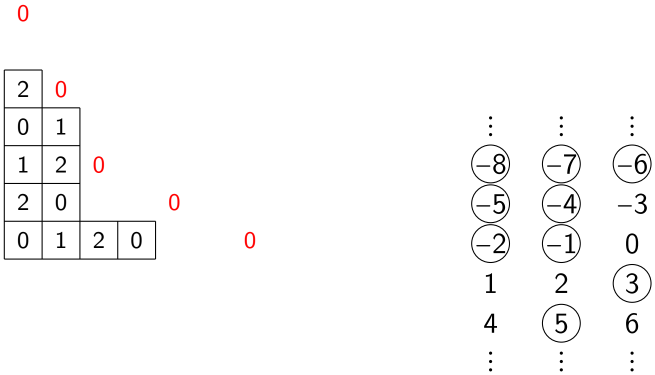

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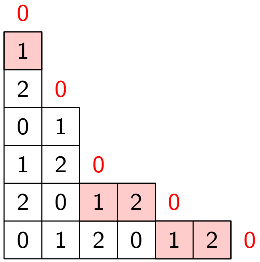

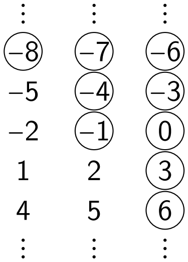

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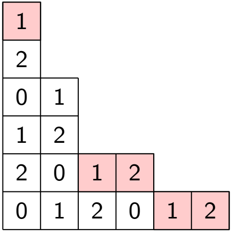
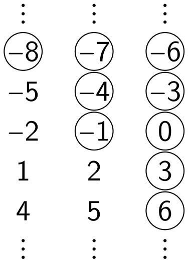




The coarse case

So p_* combinatorially corresponds to pushing all the bead to the right as much as possible. This is the map p.

Corollary

$\mathcal{Z}^0 \longleftrightarrow$ those abacus configurations where no bead can be moved to the right.

Using this it is possible to prove

Proposition

Let $\xi = \mathrm{e}^{\frac{2\pi i}{n+2}}$. Then under the substitution $q_i = \xi$ $i \in \{1, \dots, n\}$, $q_0 = \xi^{-n} q$ the terms in $Z_{[\mathbb{C}^2/G]}(q_0, \dots, q_n)$ corresponding to the preimage of a 0-generated diagram λ_0 add up to the term corresponding to λ_0 in $Z_{\mathbb{C}^2/G}(q)$. That is

$$\sum_{\mu \in p^{-1}(\lambda_0)} \underline{q}^{\mathrm{wt}(\mu)} \Big|_{q_1 = \dots = q_n = \xi, q_0 = \xi^{-n} q} = q^{\mathrm{wt}_0(\lambda_0)} .$$

The coarse generating series

Corollary

$$Z_{\mathbb{C}^2/G_\Delta}(q) =$$

$$= \left(\prod_{m=1}^{\infty} (1-q^m)^{-1} \right)^{n+1} \sum_{\overline{m}=(m_1,\dots,m_n)\in\mathbb{Z}^n} \xi^{m_1+m_2+\dots+m_n} (q^{1/2})^{\overline{m}^\top \cdot C_\Delta \cdot \overline{m}},$$

where $\xi = e^{\frac{2\pi i}{n+2}}$ *and* C *is the Cartan matrix of type* A.

The general statement

Theorem

$G \subset \mathrm{Sl}_2(\mathbb{C})$ subgroup of type A_n, D_n (conjecture for E_n), C is the finite type *Cartan matrix, h the (dual) Coxeter of the root system.*

- $$Z_{[\mathbb{C}^2/G]}(q_0, \dots, q_n) = \frac{\sum_{\overline{m}=(m_1,\dots,m_n)\in\mathbb{Z}^n} q_1^{m_1} \cdot \dots \cdot q_n^{m_n} (q^{1/2})^{\overline{m}^\top \cdot C_\Delta \cdot \overline{m}}}{\prod_{m=1}^{\infty}(1-q^m)^{n+1}},$$

  where $q = \prod_{i=0}^{n} q_i^{d_i}$ with $d_i = \dim \rho_i$, and each term corresponds to an affine cell in $\mathrm{Hilb}([\mathbb{C}^2/G])$;

- $$Z_{\mathbb{C}^2/G_\Delta}(q) = \frac{\sum_{\overline{m}=(m_1,\dots,m_n)\in\mathbb{Z}^r} \zeta^{m_1+m_2+\dots+m_n} (q^{1/2})^{\overline{m}^\top \cdot C_\Delta \cdot \overline{m}}}{\prod_{m=1}^{\infty}(1-q^m)^{n+1}},$$

  where $\zeta = e^{\frac{2\pi i}{1+h}}$, and each term corresponds to an affine cell in $\mathrm{Hilb}(\mathbb{C}^2/G)$.

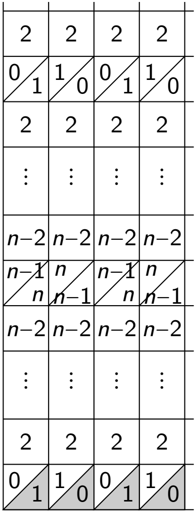
The D_n case

- Generated by

$$\sigma = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \qquad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

  where ε a fixed primitive $(2n-4)$-th root of unity.
- Bad news: there is no compatible $(\mathbb{C}^*)^2$ action.
- Good news: there is a compatible $\mathbb{C}^*$ action.
- Possible to show: the fixed points $\mathrm{Hilb}([\mathbb{C}^2/G])^{\mathbb{C}^*}$ are affine spaces parametrized by the Young walls of type D_n.
- There is an abacus description of these.
- The substitution works.

Young wall pattern of type D_n



Moduli spaces of torsion free sheaves

Points of $\mathrm{Hilb}^n(S) \longleftrightarrow$ ideal sheaves of colength n on $S \longleftrightarrow$ torsion free sheaves of rank 1, $c_1 = 0$ and $c_2 = n$.
There are analogous moduli spaces for higher rank (H-semistable) torsion-free sheaves: $M_S^H(ch)$.

Conjecture (S-duality (Vafa, Witten))

For S a smooth projective algebraic surface the generating series

$$Z_S^H(\tau) = \sum_{ch} \chi(M_S^H(ch))q^{f(ch)}$$

with $q = \mathrm{e}^{2\pi i \tau}$ should be a meromorphic modular form for some congruence subgroup of $SL_2(\mathbb{Z})$.

The S-duality conjecture - simple singularities

Theorem (S-duality for simple singularities)

For type A and type D, and, conjecturally for type E, the partition function $Z_{\mathbb{C}^2/G}(q)$ is, up to a suitable fractional power of q, the q-expansion of a meromorphic modular form of weight $-\frac{1}{2}$ for some congruence subgroup of $SL_2(\mathbb{Z})$.

Corollary (S-duality for surfaces with simple singularities)

Let X be a quasiprojective surface with simple singularities of type A and D, or, assuming the conjecture, arbitrary type. Then the partition function

$$Z_X(q) = \sum_{m=0}^{\infty} \chi\left(\mathrm{Hilb}^m(X)\right) q^m$$

is, up to a suitable fractional power of q, the q-expansion of a meromorphic modular form of weight $-\frac{\chi(X)}{2}$ for some congruence subgroup of $SL_2(\mathbb{Z})$.

Future plans

- More general (quotient) singularities
- Lift to motivic integrals
- Moduli of higher rank sheaves on singularities
- Representation theory behind $H^*(\mathrm{Hilb}(\mathbb{C}^2/G))$ (coset vertex algebras)
- ...

Thank you for your attention!

Questions?