Introduction to étale cohomology

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Outline

The site of a topological space

Let

- X be a topological space,
- X_{cl} be the set of all open subsets of X,
- $\operatorname{cov}(X_{cl})$ be the set of families $\{U_i \to U\}$ which are coverings of an $U \subseteq X$ open.

X_{cl} becomes a category if we set:

$$\operatorname{Hom}(U, V) = \begin{cases} \varnothing & \text{if } U \not\subset V \\ \text{inclusion } U \to V & \text{if } U \subset V \,. \end{cases}$$

In this category if $U_1 \to U$ and $U_2 \to U$ are arrows, then their fiber product is their intersection:

$$U_1 \times_U U_2 = U_1 \cap U_2 \,.$$

Properties of $\mathrm{cov}(X_{cl})$

Proposition

(T1) *For* $U_i \to U \in \mathrm{cov}(X_{cl})$ *and a morphism* $V \to U$ *in* X_{cl} *all fibre products* $U_i \times_U V$ *exist and* $\{U_i \times_U V \to V\} \in \mathrm{cov}(X_{cl})$.

(T2) *Given* $\{U_i \to U\} \in \mathrm{cov}(X_{cl})$ *and a family* $\{V_{ij} \to U_i\} \in \mathrm{cov}(X_{cl})$ *for all* $i \in I$, *the family* $\{V_{ij} \to U\}$ *obtained by composition of morphisms, also belongs to* $\mathrm{cov}(X_{cl})$.

(T3) *If* $V \to U$ *is an isomorphism in* X_{cl}, *then* $\{V \to U\} \in \mathrm{cov}(X_{cl})$.

In fact, the set $\mathrm{cov}(X_{cl})$ describes the topology of X.

Grothendieck topologies

Definition

A *topology* (or *site*) T consists of a category $\mathrm{cat}(T)$ and a set $\mathrm{cov}(T)$ of *coverings*, i.e. families $\{U_i \to U\}_{i \in I}$ of morphisms in $\mathrm{cat}(T)$, which satisfy (T1), (T2) and (T3).

Definition

A *morphism* $f : T \to T'$ of topologies is a functor $f : \mathrm{cat}(T) \to \mathrm{cat}(T')$ of the underlying categories with the following two properties

(a) $\{U_i \to U\} \in \mathrm{cov}(T) \Rightarrow \{f(U_i) \overset{f(\,)}{\to} f(U)\} \in \mathrm{cov}(T')$

(b) For $\{U_i \to U\} \in \mathrm{cov}(T)$ and a morphism $V \to U$ in $\mathrm{cat}(T)$ the canonical morphism

$$f(U_i \times_U V) \to f(U_i) \times_{f(U)} f(V)$$

is an isomorphism for all i.

Presheaves and sheaves on topological spaces

Let $\mathcal{C}$ be a category (e.g. $\mathcal{S}ets$ or $\mathcal{A}b$).
If X is a topological space, a presheaf on X with values in $\mathcal{C}$ is a functor

$$F : X_{cl}^{op} \to \mathcal{C} \ .$$

For every presheaf F of sets on X and every $\{U_i \to U\} \in \mathrm{cov}(T)$ there is a diagram

$$F(U) \to \prod_i F(U_i) \overset{pr_1^*}{\underset{pr_2^*}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j) \ .$$

Here $F(U) \to \prod_i F(U_i)$ is induced by the restrictions $F(U) \to F(U_i)$, and $\prod_i F(U_i) \overset{pr_1^*}{\to} \prod_{i,j} F(U_i \times_U U_j)$ is induced by $pr_1^* : F(U_i) \overset{pr_1^*}{\to} \prod_j F(U_i \times_U U_j)$ for each i (pr_2^* similarly).

The sheaf condition

The presheaf $F : X_{cl}^{op} \to \mathcal{C}$ is a sheaf, if the following holds:

(SH) For every $\{U_i \to U\} \in \mathrm{cov}(T)$ and every $a_i \in F(U_i)$, such that $pr_1^*(a_i) = pr_2^*(a_j) \in F(U_i \times_U U_j)(= F(U_i \cap U_j))$ for every i, j, there is a unique $a \in F(U)$ whose pullback to $F(U_i)$ is a_i.

Equivalently:

(SH') For every $\{U_i \to U\} \in \mathrm{cov}(T)$ the diagram

$$F(U) \to \prod_i F(U_i) \overset{pr_1^*}{\underset{pr_2^*}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j)$$

has the properties:

- $F(U) \to \prod_i F(U_i)$ is injective,
- $\mathrm{Im}(F(U) \to \prod_i F(U_i)) = \{(a_i) \in \prod_i F(U_i) | pr_1^*(a_i) = pr_2^*(a_j)\ \forall i, j\}$.

Presheaves and sheaves on sites

Let $\mathcal{C}$ be an category and T a topology.

Definition

1. A *presheaf* on X with values in $\mathcal{C}$ is a contravariant functor

$$F : T \to \mathcal{C} \,,$$

2. F is a *sheaf* if it moreover satisfies (SH), or equivalently (SH').
3. A *morphism of (pre)sheaves* $F \to G$ is a natural transformation of functors.

Abelian presheaves and sheaves on a topology T form abelian categories $\mathcal{P}$ and $\mathcal{S}$.

Sheaffication

All sheaves are presheaves, so there is an inclusion functor

$$i : \mathcal{S} \to \mathcal{P} \ .$$

Theorem

There exist a left-adjoint functor $\# : \mathcal{S} \to \mathcal{P}$ *of* i*.*

Definition

For each $F \in \mathcal{P}$, the sheaf $F^{\#}$ is called the *sheaf associated to the presheaf* F.

This is a universal construction in the sense, that each morphism from F to an abelian sheaf G factors uniquely as $F \to F^{\#} \to G$.

Refinement of coverings

Definition
$\{U'_j \to U\}_{j \in J} \to \{U_i \to U\}_{i \in I}$ if there is an $\varepsilon : J \to I$, such that $\{U'_j \to U\}$ factorizes as

$$\begin{array}{ccc} U'_j & \longrightarrow & U \\ & \searrow \quad \nearrow & \\ & U_{\varepsilon(j)} & \end{array} .$$

$\rightsquigarrow$ an inverse system of covers can be constructed.

Reminder on derived functors

- An abelian category $\mathcal{C}$ has enough injectives, if for each object A there is a monomorphism $A \to I$ into an injective object of $\mathcal{C}$.
- If $F : \mathcal{C} \to \mathcal{A}b$ is an additive, left-exact functor, then its derived functor is defined as
  1. Construct an injective resolution of X:
  $$0 \to X \to I^0 \to I^1 \to I^2 \dots .$$
  2. Apply F on it and chop off the first term:
  $$0 \to F(I^0) \to F(I^1) \to F(I^2) \dots .$$
  3. The i-th derived functor of F on X is
  $$R^i F(X) := \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1}) .$$

Cohomology of sheaves

$\mathcal{S}$ has enough injectives $\Rightarrow$ we can take right derived functors.
Consider for a fixed $U \in \mathcal{T}$ the section functor

$$\Gamma_U : \mathcal{S} \to \mathcal{A}b\,,$$

defined by $\Gamma_U(F) = F(U)$. This is additive, left-exact, and

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\Gamma_U} & \mathcal{A}b \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle \Gamma_U} & \\ & \mathcal{P} & \end{array}$$

Definition

For $q \geq 0$, the *q-th cohomology group of U with values in F* is

$$H^q(U, F) := R^q\Gamma_U(F)\,.$$

Direct/inverse images for presheaves

Let $f : T \to T'$ be a morphism of topologies, and $\mathcal{P}, \mathcal{S}$ and $\mathcal{P}', \mathcal{S}'$ be the categories of abelian (pre)sheaves on T and T', respectively.

Definition

If F' is an abelian presheaf on T', then its *direct image* $f^p F'$ is the presheaf on T given by

$$U \mapsto f^p F'(U) = F'(f(U)) \,,$$

for $U \in T$. This is functorial in F' $\rightsquigarrow$ we get an additive, exact functor:

$$f' : T' \to T \,.$$

Theorem

The functor f^p has a left adjoint f_p, which is right-exact.

Direct/inverse images for sheaves

These induce functors between $\mathcal{S}$ and $\mathcal{S}'$ as well:

1. $$f^s : \mathcal{S}' \to \mathcal{S} \ , \quad f^s = \# \circ f^p \circ i' \ ,$$
2. $$f_s : \mathcal{S} \to \mathcal{S}' \ , \quad f_s = \#' \circ f_p \circ i \ .$$

Cohomology and limits

Definition

A topology T is noetherian, if each object of T is quasi-compact.

Theorem

Assume T is noetherian, and $\mathcal{I}$ is a category with a sensible definition of limit (pseudofiltered category). Then

$$\varinjlim_{\mathcal{I}} H^q(U, F_i) \simeq H^q(U, \varinjlim_{\mathcal{I}} F_i)$$

The implicit function theorem

Theorem

If $f_1, \dots, f_k$ *are analytic functions around* $x \in \mathbb{C}^{k+n}$*, such that* $\det_{1 \leq i,j \leq k}\left(\frac{\partial f_i}{\partial x_j}\right)(x) \neq 0$*, then the projection*

$$(f_1 = \cdots = f_k = 0) \to \mathbb{C}^n$$

$$(x_1, \dots, x_{k+n}) \mapsto (x_{k+1}, \dots, x_{k+n})$$

is a local analytic isomorphism around x*.*

This is not true in the Zariski topology of AG

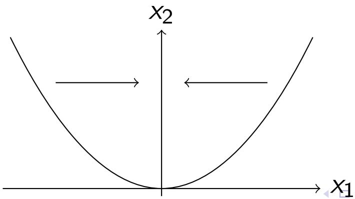
Example

$$V(x_1^2 - x_2) \to \mathbb{A}^1, \quad (x_1, x_2) \mapsto x_2 \ .$$

At $x = (1,1)$ the conditions of IFT are satisfied:

$$\frac{\partial}{\partial x_1}(x_1^2 - x_2)\ |_x = 2x_1\ |_x = 2 \neq 0\ .$$

But for all $U \subset V(x_1^2 - x_2)$ Zariski open containing x the projection to x_2 is not even a bijection: except for finitely many values of a, $(+\sqrt{a}, a), (-\sqrt{a}, a) \in U \Rightarrow a$ has two preimages.



Étale morphisms

Definition

1. The morphism $X = \mathrm{Spec} R[x_1, \dots, x_n]/(f_1, \dots, f_k) \to \mathrm{Spec} R = Y$ is *étale* in $x \in X$, if $\det_{1 \le i,j \le k} \left(\frac{\partial f_i}{\partial x_j} \right)(x) \ne 0$.
2. The finite type morphism $f : X \to Y$ is *étale*, if for all $x \in X$ there are open neighbourhoods $x \in U \subset X$ and $f(x) \in V \subset Y$ such that $F(U) \subset V$ and $f|_U$ is étale:

$$\begin{array}{ccc} U & \hookrightarrow & \mathrm{Spec} R[x_1, \dots, x_n]/(f_1, \dots, f_n) \\ \downarrow{\scriptstyle f|_U} & & \downarrow \\ V & \hookrightarrow & \mathrm{Spec} R \end{array} .$$

The étale site of a scheme

Idea: we change the topology in order for the IFT to hold.
We require that open subsets are given by étale morphisms.
$\rightsquigarrow$ we need a Grothendieck topology!

Definition

- Et/X = category of étale X-schemes
  - $\mathrm{ob}(Et/X) = \{Y \to X \text{ étale}\}$
  - $\mathrm{Hom}(Y_1 \to X, Y_2 \to X) =$

$$\left\{ \begin{array}{ccc} Y_1 & \longrightarrow & Y_2 \\ & \searrow \quad \swarrow & \\ & X & \end{array} \quad \text{commutative} \right\}$$

The étale site of a scheme

Definition

- A family $\{X_i' \stackrel{\varphi_i}{\to} X'\}$ of morphisms in Et/X is called *surjective* if $X' = \bigcup_i \varphi_i(X_i')$
- The étale site $X_{\text{ét}}$ of X:
  - $\mathrm{cat}(X_{\text{ét}}) = Et/X$,
  - $\mathrm{cov}(X_{\text{ét}})$ = set of surjective families of morphisms in Et/X.
  - Remark: these satisfy the axioms T1, T2 and T3.
- $\tilde{X}_{\text{ét}}$ = category of abelian sheaves on $X_{\text{ét}}$.

Zariski and étale cohomology

Proposition

Open immersions are étale.

Corollary

1. *Let X_{Zar} be the topology of open sets of the scheme X. Then the inclusion*

$$\varepsilon : X_{Zar} \to X_{\acute{e}t}$$

   is a morphism of topologies.

2. *By spectral sequence arguments there is a functorial morphism*

$$H^p_{Zar}(X, R^q\varepsilon^s(F)) \to H^{p+q}_{\acute{e}t}(X, F) \ ,$$

   which is in general not an isomorphism.

Equivalent conditions of étaleness

Theorem

For a morphism of schemes $f : X \to Y$ the followings are equivalent:

1. *f is étale*
2. *f is smooth and unramified*
3. *f is smooth and of relative dimension 0*
4. *f is flat, locally of finite presentation, and for every $y \in Y$, the fiber $f^{-1}(y)$ is a disjoint union of points, each of which is a finite separable field extension of the residue field $\kappa(y)$.*

Proposition

Étale morphisms are preserved under composition and base change.

Cohomology of curves

Theorem

X smooth projective algebraic curve over $\mathbb{C}$ with genus g. Then

$$H^0(X_{\mathrm{an}}, \mathbb{Z}) = \mathbb{Z} \,,$$

$$H^1(X_{\mathrm{an}}, \mathbb{Z}) = \mathbb{Z}^{2g} \,,$$

$$H^2(X_{\mathrm{an}}, \mathbb{Z}) = \mathbb{Z} \,.$$

Theorem

X smooth projective algebraic curve over k (algebraically closed) with genus g. $(\mathrm{char}\, k, n) = 1$. Then

$$H^0(X_{\acute{e}t}, \mu_n) = \mu_n(k) \,,$$

$$H^1(X_{\acute{e}t}, \mu_n) = (\mu_n(k))^{2g} \,,$$

$$H^2(X_{\acute{e}t}, \mu_n) = \mu_n(k) \,.$$

Cohomology of fields

Let $X = \mathrm{Spec}(k)$ and $G = \mathrm{Gal}(k^{\mathrm{sep}}|k)$ its absolute Galois group.

Theorem

- *$Y \to X$ is étale $\iff Y = \mathrm{Spec}(\prod_{i=1}^{r} L_i)$, where $L_i|k$ is a finite separable extension.*
- *The functor*

$$\tilde{X}_{\acute{e}t} \to [\mathit{Continous\ G\text{-}sets}]$$

$$F \mapsto \varinjlim_{k \subset k' \subset k^{\mathrm{sep}},\ \mathit{finite}} F(\mathrm{Spec}(k'))$$

  is an equivalence of categories.
- $$H^q(X_{\acute{e}t}, F) \cong H^q(G, \varinjlim_{k'} F(\mathrm{Spec}(k')))$$
- *Here the right-hand side is the Galois-cohomology.*

l-adic comology

Étale cohomology yields the right cohomology theory for *torsion coefficients*.
More effort is needed for coefficients in a field with characteristic 0
$\rightsquigarrow$ l-adic cohomology ($l \neq \mathrm{char}\, k$ prime):

$$H^i(X, \mathbb{Z}_l) = \varprojlim_{\nu} H^i(X_{\text{ét}}, \mathbb{Z}/l^\nu\mathbb{Z}) \,,$$

$$H^i(X, \mathbb{Q}_l) = H^i(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \,.$$

Properties of l-adic comology

Theorem

1. *The groups $H^i(X, \mathbb{Q}_l)$ are vector spaces over $\mathbb{Q}_l$.*
2. *If X is proper over k, then they are finite dimensional.*
3. *Functoriality in X: if $f : X \to Y$ is a morphism, then it induces a homomorphism on the cohomologies:*

$$f^* : H^i(Y, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l) .$$

4. $H^i(X, \mathbb{Q}_l) = 0$ *for* $i > 2 \dim X$.
5. *Künneth-formula is valid.*

Properties of l-adic comology

Theorem

6. *There is a cup-product structure*

$$H^i(X, \mathbb{Q}_l) \times H^j(X, \mathbb{Q}_l) \to H^{i+j}(X, \mathbb{Q}_l)$$

   defined for all i, j.

7. *Poincaré duality: if* X *is smooth and proper over* k*, of dimension* n*, then* $H^{2n}(X, \mathbb{Q}_l)$ *is 1-dimensional, and the cup-product pairing is a perfect pairing for each* i*,* $0 \le i \le 2n$*.*

Lefschetz fixed-point formula

Theorem

Let X be smooth and proper over k. Suppose $f : X \to X$ has only isolated fixed points, whose number is $L(f, X)$. Assume moreover, that for each fixed point $x \in X$, assume that the action of $1 - \mathrm{d}f$ on Ω^1_X is injective. Then

$$L(f, X) = \sum_{i=0}^{2n} (-1)^i \mathrm{Tr}(f^* H^i(X, \mathbb{Q}_l)) \, .$$

Thank you for your attention!

Questions?