

Euler Characteristics of Hilbert Schemes of Points On Surfaces with Simple Singularities

Ádám Gyenge^{1,*}, András Némethi¹, and Balázs Szendrői²

¹Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, H-1053, Budapest, Hungary and ²Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

*Correspondence to be sent to: e-mail: gyenge.adam@renyi.mta.hu

This is an announcement of conjectures and results concerning the generating series of Euler characteristics of Hilbert schemes of points on surfaces with simple (Kleinian) singularities. For a quotient surface \mathbb{C}^2/G with $G < SL(2, \mathbb{C})$ a finite subgroup, we conjecture a formula for this generating series in terms of Lie-theoretic data, which is compatible with existing results for type A singularities. We announce a proof of our conjecture for singularities of type D . The generating series in our conjecture can be seen as a specialized character of the basic representation of the corresponding (extended) affine Lie algebra; we discuss possible representation-theoretic consequences of this fact. Our results, respectively conjectures, imply the modularity of the generating function for surfaces with type A and type D , respectively arbitrary, simple singularities, confirming predictions of S -duality.

1 Euler Characteristics of Hilbert Schemes of Points

Let X be a quasiprojective variety X over the field \mathbb{C} of complex numbers. Let $\text{Hilb}^m(X)$ denote the Hilbert scheme of m points on X , the quasiprojective scheme parametrizing zero-dimensional subschemes of X of length m . Consider the generating series of

Received January 8, 2016; Revised January 8, 2016; Accepted May 31, 2016

topological Euler characteristics

$$Z_X(q) = \sum_{m=0}^{\infty} \chi(\text{Hilb}^m(X)) q^m.$$

For a smooth variety X , the series $Z_X(q)$, as well as various refinements, have been extensively studied. For a nonsingular curve $X = C$, we have MacDONALD's result [8]

$$Z_C(q) = (1 - q)^{-\chi(C)}.$$

For a nonsingular surface $X = S$, we have (a specialization of) GÖTTSCHE's formula [5]

$$Z_S(q) = \left(\prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{\chi(S)}. \quad (1)$$

There are also results for higher-dimensional varieties [1].

For singular varieties X , very little is known about the series $Z_X(q)$. For a singular curve $X = C$ with a finite set $\{P_1, \dots, P_k\}$ of planar singularities however, we have the beautiful conjecture of Oblomkov and Shende [14], proved by Maulik [9], which specializes to the following:

$$Z_C(q) = (1 - q)^{-\chi(C)} \prod_{j=1}^k Z^{(P_j, C)}(q). \quad (2)$$

Here each $Z^{(P_i, C)}(q)$ is a highly nontrivial local term that can be expressed in terms of the HOMFLY polynomial of the embedded link of the singularity $P_i \in C$.

2 Simple Surface Singularities

In this announcement, we consider the generating series $Z_S(q)$ for $X = S$ a singular surface with simple (Kleinian, rational double point) singularities. First, we discuss the local situation. As is well known, locally analytically S is a quotient singularity $S = \mathbb{C}^2/G_\Delta$. Here $G_\Delta < \text{SL}(2, \mathbb{C})$ is a finite subgroup corresponding to an irreducible simply-laced Dynkin diagram Δ , the dual graph of the exceptional components in the minimal resolution of the singularity. There are three possible types: Δ can be of type A_n for $n \geq 1$, type D_n for $n \geq 4$, and type E_n for $n = 6, 7, 8$. The following is our main conjecture.

Conjecture 2.1. Let \mathbb{C}^2/G_Δ be a simple singularity associated with the irreducible simply-laced Dynkin diagram Δ with n nodes. Let C_Δ be the Cartan matrix corresponding to Δ , and let h^\vee be the (dual) Coxeter number of the corresponding finite-dimensional Lie algebra. Then

$$Z_{\mathbb{C}^2/G_\Delta}(q) = \left(\prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{n+1} \cdot \sum_{\bar{m}=(m_1, \dots, m_n) \in \mathbb{Z}^n} \zeta^{m_1+m_2+\dots+m_n} (q^{1/2})^{\bar{m}^\top \cdot C_\Delta \cdot \bar{m}}, \quad (3)$$

where $\zeta = \exp(\frac{2\pi i}{1+h^\vee})$. □

Evidence for the conjecture is presented in the following result.

Theorem 2.2. Let $S = \mathbb{C}^2/G_\Delta$ be a simple singularity with Δ of type A_n for $n \geq 1$ or D_n for $n \geq 4$. Then Conjecture 2.1 holds. □

Proof. For type A , straightforward torus localization leads to a combinatorial series. This series was determined in closed form by Dijkgraaf and Sułkowski [2] using a direct method, obtaining formula [2, (10)], which is equivalent to (3) for type A . The same result, still for type A , was recently re-proved, using wall-crossing in Donaldson–Thomas theory, by Toda [16]. We give a third, elementary combinatorial argument in [7, Section 2].

For type D , formula (3) is the main result of [7]. ■

Remark 2.3. A priori it is not clear at all (at least to us) that the right hand side of (3) defines an integer series. We checked numerically that the series has integer coefficients for Δ of type E_6, E_7 , and E_8 as well to a high power in q . See Section 5 for further discussion. We also note that while Theorem 2.2 covers in some sense “most” of the cases, the existing proofs are specific to types A and D and it seems difficult to push them through in type E . □

3 Surfaces with Simple Singularities

Let $X = S$ be a quasiprojective surface which is nonsingular outside a finite number of simple surface singularities $\{P_1, \dots, P_k\}$, with $(P_i \in S)$ a singularity locally analytically isomorphic to $(0 \in \mathbb{C}^2/G_{\Delta_i})$ for $G_{\Delta_i} < \text{SL}(2, \mathbb{C})$ a finite subgroup as above. Let $S^0 \subset S$ be the nonsingular part of S .

Theorem 3.1. The generating function $Z_S(q)$ of the Euler characteristics of Hilbert schemes of points of S has a product decomposition

$$Z_S(q) = \left(\prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{\chi(S^0)} \cdot \prod_{j=1}^k Z^{(P_j, S)}(q). \quad (4)$$

The local terms can be expressed as

$$Z^{(P_i, S)}(q) = Z_{\mathbb{C}^2/G_{\Delta_i}}(q) \quad (5)$$

and are given by formula (3) for $P_i \in S$ of type A and D , and, assuming Conjecture 2.1, also of type E . \square

Proof. The product decomposition (4), as well as the equality (5), follow from a standard argument; we sketch the details. For a point $P \in S$ on a quasiprojective surface S , let $\text{Hilb}_P^m(S)$ denote the punctual Hilbert scheme of S at P , the Hilbert scheme of length m subschemes of S set-theoretically supported at the single point P . Then for our surface S , we have

$$\chi(\text{Hilb}_P^m(S)) = \chi(\text{Hilb}_0^m(\mathbb{C}^2/G_{\Delta_i})) = \chi(\text{Hilb}^m(\mathbb{C}^2/G_{\Delta_i})).$$

Here the first equality follows from the analytic isomorphism between $(P_i \in S)$ and $(0 \in \mathbb{C}^2/G_{\Delta_i})$. The second equality follows from torus localization, using the fact that each singularity $\mathbb{C}^2/G_{\Delta_i}$ is weighted homogeneous, admitting a \mathbb{C}^* -action fixing only the origin. On the other hand, we have the decomposition

$$\text{Hilb}^m(S) = \bigsqcup_{\sum_{i=0}^k m_i = m} \text{Hilb}^{m_0}(S^0) \times \prod_{i=1}^k \text{Hilb}_{P_i}^{m_i}(S).$$

Reinterpreting this equality for generating series proves (4)–(5), recalling also (1) for S^0 . The last part of the statement is Theorem 2.2. \blacksquare

Formulae (3)–(5) are our analogue for the case of surfaces with simple singularities of the Oblomkov–Shende–Maulik formula (2). Note that each $\mathbb{C}^2/G_{\Delta_i}$ is in particular a hypersurface singularity, as are planar singularities in the curve case. The main difference with formula (2) is the fact that (conjecturally, for type E) our local terms $Z^{(P_i, S)}(q)$ are expressed in terms of Lie-theoretic and not topological data. We leave the question whether our local terms have any interpretation of in terms of the topology of the

(embedded) link of P_i , and whether there are nice formulas for other two-dimensional hypersurface singularities, for further work.

4 Modularity Results

Our formulae lead to the following new modularity results, extending the results of [16] for type A .

Corollary 4.1 (S-duality for simple singularities). For type A and type D , and, assuming Conjecture 2.1, for all types, the partition function $Z_{\mathbb{C}^2/G_\Delta}(q)$ is, up to a suitable fractional power of q , the q -expansion of a meromorphic modular form of weight $-\frac{1}{2}$ for some congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$. \square

Proof. This follows straight from [16, Proposition 3.2]. \blacksquare

Corollary 4.2 (S-duality for surfaces with simple singularities). Let S be a quasiprojective surface with simple singularities of type A and D , or, assuming Conjecture 2.1, of arbitrary type. Then the generating function $Z_S(q)$ is, up to a suitable fractional power of q , the q -expansion of a meromorphic modular form of weight $-\frac{\chi(S)}{2}$ for some congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$. \square

Proof. Combine Theorem 3.1 and Corollary 4.1. \blacksquare

5 A Representation of an Affine Lie Algebra

Our formula (3) has a strong representation-theoretic flavour. We discuss the relevant ideas in this section, and conclude with some speculations.

Let \mathfrak{g}_Δ be the complex finite dimensional simple Lie algebra of rank n corresponding to the irreducible simply-laced Dynkin diagram Δ with n nodes. Attached to Δ is also an (untwisted) affine Lie algebra $\widetilde{\mathfrak{g}}_\Delta$; a slight variant will be more interesting for us, see for example, [3, Section 6]. Denote by $\widetilde{\mathfrak{g}}_\Delta \oplus \mathbb{C}$ the Lie algebra that is the direct sum of the affine Lie algebra $\widetilde{\mathfrak{g}}_\Delta$ and an infinite Heisenberg algebra \mathfrak{heis} , with their centres identified; we may call $\widetilde{\mathfrak{g}}_\Delta \oplus \mathbb{C}$ the extended affine Lie algebra associated with Δ .

Let V_0 be the basic representation of $\widetilde{\mathfrak{g}}_\Delta$, the level-1 representation with highest weight ω_0 , the fundamental weight corresponding to the additional node of the affine Dynkin diagram. Let \mathcal{F} be the standard Fock space representation of \mathfrak{heis} , having central charge 1. Then $V = V_0 \otimes \mathcal{F}$ is a representation of $\widetilde{\mathfrak{g}}_\Delta \oplus \mathbb{C}$ that we may call the extended basic representation.

Example 5.1. For Δ of type A_n , we have $\mathfrak{g}_\Delta = \mathfrak{sl}_{n+1}$, $\tilde{\mathfrak{g}}_\Delta = \tilde{\mathfrak{sl}}_{n+1}$, and $\widetilde{\mathfrak{g}_\Delta \oplus \mathbb{C}} = \tilde{\mathfrak{gl}}_{n+1}$. In this case there is in fact a natural vector space isomorphism $V \cong \mathcal{F}$ with Fock space itself, see for example, [15, Section 3E]. \square

By the Frenkel–Kac theorem [4] there is an isomorphism

$$V \cong \mathcal{F}^{n+1} \otimes \mathbb{C}[Q_\Delta],$$

where $(Q_\Delta, \langle \cdot \rangle)$ is the root lattice corresponding to the root system Δ . Here, for $\beta \in Q_\Delta$, $\mathcal{F}^{n+1} \otimes e^\beta$ is the sum of weight subspaces of weight $\omega_0 - (m + \frac{\langle \beta, \beta \rangle}{2})\delta + \beta$, $m \geq 0$, where δ is the imaginary root. Thus, we can write the character of the representation V as

$$\text{char}_V(q_0, \dots, q_n) = e^{\omega_0} \left(\prod_{m>0} (1 - q^m)^{-1} \right)^{n+1} \cdot \sum_{\beta \in Q_\Delta} q_1^{\beta_1} \dots q_n^{\beta_n} (q^{1/2})^{\langle \beta, \beta \rangle}, \tag{6}$$

where $q = e^{-\delta}$, and $\beta = (\beta_1, \dots, \beta_n) \in Q_\Delta$ is the expression of an element of the root lattice Q_Δ in terms of the simple roots.

Remark 5.2. The representation V itself is of course well known to appear in the equivariant geometry of the pair (\mathbb{C}^2, G_Δ) . Given any finite-dimensional representation $\rho \in \text{Rep}(G_\Delta)$, denote by $\text{Hilb}^\rho(\mathbb{C}^2)$ the ρ -Hilbert scheme of \mathbb{C}^2 , the scheme representing G_Δ -equivariant ideals $I \triangleleft \mathbb{C}[\mathbb{C}^2]$ with $\mathbb{C}[\mathbb{C}^2]/I \cong \rho$. These are all Nakajima quiver varieties for the affine Dynkin diagram associated with Δ (the McKay quiver of G_Δ). By results of Nakajima [10, 13], there is in fact an isomorphism of $\widetilde{\mathfrak{g}_\Delta \oplus \mathbb{C}}$ -representations

$$V \cong \bigoplus_{\rho \in \text{Rep}(G_\Delta)} H^*(\text{Hilb}^\rho(\mathbb{C}^2)),$$

where the $\widetilde{\mathfrak{g}_\Delta \oplus \mathbb{C}}$ -action on the right hand side is given by natural correspondences. Since these varieties have no odd cohomology [12, Section 7], the Euler characteristic generating series of the right hand side is given precisely by the character (6) of V . \square

Comparing formulas (3) and (6), we deduce the following.

Theorem 5.3. For Δ of type A and D , or, assuming Conjecture 2.1, of arbitrary type, the generating series $Z_{\mathbb{C}^2/G_\Delta}(q)$ is a specialization of the character formula of the extended basic representation of the associated extended affine Lie algebra $\widetilde{\mathfrak{g}_\Delta \oplus \mathbb{C}}$, obtained by setting $q_i = \exp(\frac{2\pi i}{1+h^\vee})$ for $i = 1, \dots, n$. \square

We remark here that Dijkgraaf and Sułkowski [2] already noticed the fact that (3) for type A is a specialized character formula for $\widetilde{\mathfrak{sl}}_{n+1}$.

We do not know whether (3) itself is a character formula for a Lie algebra, but this appears plausible. Such a result would of course imply in particular that its coefficients are (positive) integers. It would also suggest that suitable cohomology groups $\oplus_m H^*(\text{Hilb}^m(\mathbb{C}^2/G_\Delta))$, and indeed $\oplus_m H^*(\text{Hilb}^m(S))$ for S a surface with simple singularities, would carry actions of interesting Lie algebras, generalizing the Heisenberg algebra actions of Grojnowski and Nakajima [6, 11] when S is a smooth surface.

In a different direction, one may also wonder about a higher-rank generalization. Our current discussion involves Hilbert schemes, parametrizing rank $r = 1$ sheaves on the singular surface. In the relationship between the instantons on algebraic surfaces and affine Lie algebras, level equals rank [6]. Indeed the extended basic representation V has level $l = 1$. Thus the substitution above is by the root of unity $\zeta = \exp(\frac{2\pi i}{l+h^\vee})$. In higher rank, one may wonder whether there are similar formulae involving Euler characteristics of degenerate versions of the moduli space of rank $r = l$ framed G_Δ -equivariant vector bundles on \mathbb{P}^2 on the one hand, and specialized character formulae of level l representations of affine Lie algebras on the other.

To conclude, we point out that the substitution by the root of unity $\zeta = \exp(\frac{2\pi i}{l+h^\vee})$ appears elsewhere in representation theory, notably in the Verlinde formula, and in the Kazhdan–Lusztig equivalence between certain categories of representations of the affine Lie algebra, respectively of the finite quantum group. We are unaware of any connection between our work and those circles of ideas.

Funding

This work was partially supported by the *Lendület program* (Momentum Programme) of the Hungarian Academy of Sciences and by ERC Advanced Grant LDTBud (awarded to András Stipsicz) to Á.Gy.; OTKA Grants 100796 and K112735 to A.N.; EPSRC Programme Grant EP/I033343/1 to B.Sz.

Acknowledgments

The authors would like to thank Gwyn Bellamy, Alexander Braverman, Alastair Craw, Eugene Gorsky, Ian Grojnowski, Kevin McGerty, Iain Gordon, Tomas Nevins, Travis Schendler, and Tamás Szamuely for helpful comments and discussions.

References

- [1] Cheah, J. "On the cohomology of Hilbert schemes of points." *Journal of Algebraic Geometry* 5, no. 3 (1996): 479–512.

- [2] Dijkgraaf, R. and P. Sułkowski. "Instantons on ALE spaces and orbifold partitions." *Journal of High Energy Physics* 3 (2008): 013.
- [3] Etingof, P. "Symplectic reflection algebras and affine Lie algebras." *Moscow Mathematical Journal* 12, no. 3 (2012): 543–65.
- [4] Frenkel, I. B. and V. G. Kac. "Basic representations of affine Lie algebras and dual resonance models." *Inventiones Mathematicae* 62, no. 1 (1980): 23–66.
- [5] Göttsche, L. "The Betti numbers of the Hilbert scheme of points on a smooth projective surface." *Mathematische Annalen* 286, no. 1 (1990): 193–207.
- [6] Grojnowski, I. "Instantons and affine algebras I: the Hilbert scheme and vertex operators." *Mathematical Research Letters* 3, no. 2 (1996): 275–92.
- [7] Gyenge, Á., A. Némethi, and B. Szendrői. *Euler characteristics of Hilbert schemes of points on simple surface singularities* (2015): <http://arxiv.org/abs/1512.06844> (accessed June 10, 2016).
- [8] Macdonald, I. G. "The Poincaré polynomial of a symmetric product." *Mathematical Proceedings of the Cambridge Philosophical Society* 58, no. 4 (1962): 563–68.
- [9] Maulik, D. "Stable pairs and the HOMFLY polynomial." *Inventiones Mathematicae* 204, no. 3 (2016): 787–831.
- [10] Nakajima, H. "Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras." *Duke Mathematical Journal* 76, no. 2 (1994): 365–416.
- [11] Nakajima, H. "Heisenberg algebra and Hilbert schemes of points on projective surfaces." *Annals of Mathematics (2)* 145, no. 2 (1997): 379–88.
- [12] Nakajima, H. "Quiver varieties and finite dimensional representations of quantum affine algebras." *Journal of the American Mathematical Society* 14, no. 1 (2001): 145–238.
- [13] Nakajima, H. Geometric Construction of Representations of Affine Algebras. In *Proceedings of the International Congress of Mathematicians (Beijing, 2002)*, edited by L. Tatsien, C. Zhijie, X. Mi and Z. Chunlian, vol. 1, 423–38. IMU, Singapore: World Scientific, 2002.
- [14] Oblomkov, A. and V. Shende. "The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link." *Duke Mathematical Journal* 161, no. 7 (2012): 1277–303.
- [15] Tingley, P. *Notes on Fock Space*. (2011): http://webpages.math.luc.edu/~ptingley/lecturenotes/Fock_space-2010.pdf (accessed June 10, 2016).
- [16] Toda, Y. "S-Duality for surfaces with A_n -type singularities." *Mathematische Annalen* 363 (2015): 679–99.