Hilbert schemes of points on some classes of surface singularities

A Ph.D. dissertation by

ÁDÁM GYENGE

submitted to

Eötvös Loránd University
Institute of Mathematics

Doctoral School:	Mathematics
Director:	Miklós Laczkovich, D.Sc.
	Professor, Member of the Hungarian Academy of Sciences
Doctoral Program:	Pure Mathematics
Director:	András Szűcs, D.Sc.
	Professor, Member of the Hungarian Academy of Sciences
Supervisor:	András Némethi, D.Sc.
	Professor, Doctor of the Hungarian Academy of Sciences
Consultant:	Balázs Szendrői, Ph.D.
	Professor, University of Oxford

2016

"Don't Panic."
– Douglas Adams, The Hitchhiker's Guide to the Galaxy

Acknowledgement

Completing the PhD and writing this thesis was an amazing journey that would not have been possible without the support and encouragement of many outstanding people.

My greatest appreciation and gratitude goes to my advisors. Balázs Szendrői helped a lot in developing the material of the thesis as well as in making it much more readable than it could be without him. András Némethi pointed out many mistakes in earlier versions and suggested several improvements. My gratitude extends to Alastair Craw, Eugene Gorsky and Tamás Szamuely for helpful comments and discussions about particular problems related to the thesis.

It was a pleasure and a great honour to work at two excellent mathematical schools in Budapest. The environments of both the ELTE and the Rényi Insitute were fantastic and motivating. I am grateful for this to Balázs Csikós and András Stipsicz respectively.

Among the unnumerous outstanding teachers during my university years I am especially thankful to Gábor Etesi, András Szűcs, Szilárd Szabó, László Fehér and Gyula Lakos.

Moreover, I would like to thank to a variety of young mathematicians who also helped in several smaller or bigger questions during my work, or with whom we just had a lot of fun and many interesting conversations. Among many others this includes József Bodnár, Gergő Pintér, Levente Nagy, Norbert Pintye, Tamás László, Marco Golla, Stefan Behrens and Baldur Sigurdsson.

Finally, I would like to express my deepest gratitude to my family: Mom, Dad, Zsuzsi, Ákos and especially to my beloved Luca and Ági (who had to suffer the most from the long evenings and weekends while I was finishing the thesis). They all stood by me and shared with me both the great and the difficult moments of life.

Contents

CHAPTER 1

Introduction

This chapter presents the motivations behind the study of Hilbert schemes of points on surface singularities and sets the aim of the research. Finally the structure of the thesis is briefly summarized.

1.1. Problem setting and background motivation

The punctual Hilbert scheme parameterizing the zero-dimensional subschemes of a quasi-projective variety contains a large amount of information about the geometry and topology of the base variety. Moreover, it is a very important moduli space. As a set, it consists of ideal sheaves of the sheaf of regular functions on the variety, such that the quotient by the ideal has finite length. The Hilbert schemes of points on smooth curves and surfaces have been investigated for a long time by several people, including Hartshorne [30], Fogarty [16], Macdonald [47], Iarrobino [34], Briançon [5]. Due to the work of Nakajima [52], Grojnowski [23], and many others, it has turned out that the surface case has an especially rich geometrical structure, see e.g. [53].

The punctual Hilbert scheme of a smooth curve or surface is a smooth variety partitioned naturally according to the length. For a smooth curve C the expression

$$\sum_{n \geq 0} \chi(\text{Hilb}^n(C))q^n = \frac{1}{(1-q)^{\chi(C)}}$$

for the generating series collecting the Euler characteristics of the different components is a corollary of the Macdonald formula [47]. Using the Weil conjecture (as proved by Deligne) Göttsche proved the following remarkable product formula for the generating series of the Euler characteristics of the Hilbert scheme of points on a smooth surface S [21]:

$$\sum_{n \geq 0} \chi(\text{Hilb}^n(S))q^n = \prod_{n \geq 1} \frac{1}{(1-q^n)^{\chi(S)}} \ .$$

In the recent years a new direction has emerged, which also allows singularities on the base variety. A breakthrough result was obtained by Maulik [48], who proved the conjecture of Oblomkov and Shende relating an integral of the length function with respect to the Euler characteristic over the Hilbert scheme of points of a curve with planar singularities to the HOMFLY polynomial of its link. The result shows that this polynomial invariant of the link of a singularity contains information about the invariants of the Hilbert scheme of points on the singularity.

Motivated by these results it is natural to consider the Hilbert scheme of points on singular surfaces. The aim of this thesis is to describe the Euler characteristics of the Hilbert schemes parameterizing the zero-dimensional subschemes of of some basic classes of surface singularities.

The well known simple singularities are the simplest type of normal surface singularities, and it is known that they have an orbifold structure. There are at least two natural versions of the punctual Hilbert scheme in the case of quotient singularities. One of these, the equivariant Hilbert scheme of $\mathbb{C}^2$ with respect to the groups of ADE type were investigated by Nakajima [51]. The other type, the coarse Hilbert scheme is more mysterious. We show that the generating series of the Euler characteristics of the coarse Hilbert schemes of points on the singularities of type A_n and D_n can be computed from the multivariable generating series of the corresponding equivariant Hilbert schemes. The remaining cases E_6, E_7 and E_8 are not treated here, but computer calculations lead us to an analogous conjecture. The proofs might be similar as well, once the representation theoretic tools become available.

The moduli spaces of torsion free sheaves on surfaces are higher rank analogs of the Hilbert schemes. In type A our results reveal their Euler characteristic generating function as well. Another, very interesting class of normal surface singularities is the so-called cyclic quotient singularities of type $(p, 1)$. As an outlook we also obtain some results about the associated generating functions.

1.2. The structure of the thesis

The thesis summarizes the results of the papers [26], [27], [28], [29], the last two of which are joint manuscripts with András Némethi and Balázs Szendrői.

The thesis can be divided into two main parts. The first part contains the preliminaries, statement of all the major results, and the proofs for the type A and the type $(p, 1)$ cyclic quotient cases.

- **Chapter 2** contains the preliminaries for the whole thesis. We summarize the necessary background and introduce the two types of Hilbert schemes which will be our central objects. The quiver description of the moduli space of higher rank torsion-free sheaves is also summarized, together with the necessary background on representation theory. Parts of this chapter have appeared in [29] and [28].
- **Chapter 3** announces the major results of the thesis as well as their corollaries related to the S-duality conjecture. These appear in [26], [29] and [27].
- In **Chapter 4** we first investigate the type A case from several viewpoints. We give a new proof to Theorem 3.2 following [26]. Second, we establish a connection between the two types of Hilbert schemes and prove Theorem 3.5.

Individually, both Theorem 3.2 and Theorem 3.5 were already known in the literature. But the unified treatment using the so-called abacus combinatorics has the advantage that it generelizes away from type A case. The treatment of this second path is based on [28]. As an outlook, we prove the results about the cyclic quotient singularities of type $(p, 1)$ as in [27].

The second part is based entirely on [28], and contains the proofs for the type D case.

- In **Chapter 5**, we introduce Schubert-style cell decompositions of Grassmannians of homogeneous summands of $\mathbb{C}[x, y]$.
- In **Chapter 6** we give a cell decomposition of the orbifold Hilbert scheme, proving Theorem 3.2 in type D.
- In **Chapter 7**, we discuss some special subsets of the strata and their geometry.
- A decomposition of the coarse Hilbert scheme is given in **Chapter 8**.
- In **Chapter 9**, the proof of Theorem 3.5 in type D is completed using combinatorial enumeration.

CHAPTER 2

Preliminaries

This chapter contains the necessary background for the thesis. We introduce the Hilbert scheme of points and its higher rank analog in the equivariant setting with respect to a finite group action. We mention their connection to quiver varieties and to the representation theory of Lie algebras of affine type.

2.1. Hilbert scheme of points and related moduli spaces

For a general reference on Hilbert scheme of points on surfaces, we refer the reader to [53]. Here we recall the basic facts.

Let X be a projective scheme over an algebraically closed field k and $\mathcal{O}_X(1)$ a very ample line bundle on X. The contravariant Hilbert functor

$$\mathcal{H}ilb(X) \colon [\text{Schemes}] \to [\text{Sets}]$$

from the category of schemes to the category of sets is defined as

$$\mathcal{H}ilb(X)(U) = \{Z \subset X \times U : Z \text{ is a closed subscheme}, \pi : Z \hookrightarrow X \times U \to U \text{ is flat}\}.$$

This means that the functor $\mathcal{H}ilb(X)$ associates to a scheme U the set of families of closed subschemes on X which are parameterized by U.

For $u \in U$, the Hilbert polynomial in u is defined as

$$P_u(k) = \chi(\mathcal{O}_{Z_u} \otimes O_X(k)),$$

where $Z_u = \pi^{-1}(u)$. By the flatness of Z over U, P_u is constant on the connected components of U. For each polynomial P let $\mathcal{H}ilb^P(X)$ be the subfunctor of $\mathcal{H}ilb(X)$ which associates to U the set of families of closed subschemes in X which has P as its Hilbert polynomial.

THEOREM 2.1 ([24]). *The functor $\mathcal{H}ilb^P(X)$ is representable by a projective scheme $\mathrm{Hilb}^P(X)$. In particular, there exists a universal family $\mathcal{Z}$ over $\mathrm{Hilb}^P(X)$, and that every family on U is induced by a unique morphism $\phi : U \to \mathrm{Hilb}^P(X)$.*

If there is an open subscheme Y of X, then there exists the corresponding open subscheme $\mathrm{Hilb}^P(Y)$ of $\mathrm{Hilb}^P(X)$ parameterizing subschemes in Y. In particular, $\mathrm{Hilb}^P(Y)$ is defined for a quasi-projective scheme Y as well.

DEFINITION 2.2. For some fixed integer $m \in \mathbb{Z}$, let P be the constant Hilbert polynomial given by $P(k) = m$ for all $k \in \mathbb{Z}$. We denote by $\mathrm{Hilb}^m(X)$ the corresponding Hilbert scheme. It is called the *Hilbert scheme of m points on X*, or the *punctual Hilbert scheme of X* if m is not important or specified previously.

As a set, $\mathrm{Hilb}^m(X)$ consists of sheaf of ideals I of the structure sheaf $\mathcal{O}_X$ of X, such that $\mathcal{O}_X/I$ is a finite length $\mathcal{O}_X$-module with $H^0(\mathcal{O}_X/I)$ an m-dimensional $\mathbb{C}$-vector space. For a locally closed subvariety $Y \subset X$, let $\mathrm{Hilb}^m(X, Y) \subset \mathrm{Hilb}^m(X)$ be the Hilbert scheme of zero-dimensional subschemes of X of length m supported set-theoretically at the points of Y. The disjoint union of these spaces for all m is denoted as $\mathrm{Hilb}(X)$, and $\mathrm{Hilb}(X, Y)$ respectively.

Let us fix an arbitrary quasi-projective variety X. A central invariant in this thesis will be the generating series of the Euler characteristics of its Hilbert schemes:

$$(2.1) \qquad Z_X(q) = \sum_{m=0}^{\infty} \chi\left(\mathrm{Hilb}^m(X)\right) q^m.$$

There is a relation between the Hilbert scheme of m points to the m-th symmetric product of X given by *Hilbert-Chow morphism*:

$$\begin{aligned} \pi \colon \mathrm{Hilb}^m(X) &\rightarrow S^m X \\ I &\mapsto \sum_{x \in X} \mathrm{colength}(I_x)[x]. \end{aligned}$$

It is known that if X is a nonsingular curve then $\mathrm{Hilb}^m(X) = S^m X$. Then we have MacDonald's result [47] for the generating series:

$$Z_C(q) = (1-q)^{-\chi(C)}.$$

Another well investigated case is when X is a smooth surface. Then the following theorem holds:

THEOREM 2.3 (Fogarty).   *(1)* $\mathrm{Hilb}^m(X)$ *is smooth of dimension* $2m$.
 (2) The morphism $\pi \colon \mathrm{Hilb}^m(X) \to S^m X$ *is a resolution of singularities.*

For any nonsingular surface S, we have (a specialization of) Göttsche's formula [21]

$$(2.2) \qquad Z_S(q) = \left(\prod_{m=1}^{\infty}(1-q^m)^{-1}\right)^{\chi(S)}.$$

For results valid for higher dimensional varieties, see [8].

A particular case of the general construction above is when $X = \mathbb{C}^2$. $\mathrm{Hilb}(\mathbb{C}^2)$ has a rich geometric structure and will play a crucial rule throughout the thesis.

For singular varieties X, the series $Z_X(q)$ is much less studied. For a singular curve C with a finite set of singuliarities $\{P_1, \dots, P_k\}$, and under the assumption that (C, p_i) is planar for each i, we have the beautiful conjecture of Oblomkov and

Shende [57], proved by Maulik [48], which takes the form

$$(2.3) \qquad Z_C(q) = (1 - q)^{-\chi(C)} \prod_{j=1}^{k} Z^{(P_i,C)}(q).$$

Here $Z^{(P_i,C)}(q)$ are highly nontrivial local terms that depend only on the embedded topological type of the link of the singularity $P_i \in C$.

The higher rank analog of the Hilbert scheme of points on $\mathbb{C}^2$ is the moduli space of framed torsion free sheaves on $\mathbb{P}^2$. Torsion free sheaves are generalizations of vector bundles, essentially they can be viewed as vector bundles which are allowed to have some singularities: the dimensions of the fibers do not all have to be equal. Their moduli space is defined as

$$\mathcal{M}^{r,m}(\mathbb{C}^2) = \left\{ (E, \Phi) \;\middle|\; \begin{array}{c} E \text{ is a torsion free sheaf of rank } r, \\ c_2(E) = m \text{ which is locally free in a} \\ \text{neighbourhood of } l_\infty, \\ \Phi \colon E|_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{l_\infty}^{\oplus r} \text{ is a framing at infinity} \end{array} \right\} \Big/ \text{ isomorphism,}$$

where $l_\infty = \{[0 : z_1 : z_2] \in \mathbb{P}^2\} \subset \mathbb{P}^2$ is the line at infinity. We put $\mathbb{C}^2$ instead of $\mathbb{P}^2$ in the argument to keep the analogy with the Hilbert scheme, but this should not eventuate any confusion. By the existence of a framing Φ, $c_1(E) = 0$ for each representative (E, Φ) of an element in $\mathcal{M}^{r,m}(\mathbb{C}^2)$.

LEMMA 2.4. *Let $H = E|_{\mathbb{C}^2}$ be a torsion free sheaf on $\mathbb{C}^2$. Then H is the subsheaf of a free sheaf in a canonical way.*

PROOF. The double dual sheaf $H^{\vee\vee}$ is reflexive. Consequently, it has depth 2. By the Auslander-Buchsbaum formula, the projective dimension of $H^{\vee\vee}$ is 0, i.e. $H^{\vee\vee}$ is projective. This implies, by the Quillen-Suslin theorem, that $H^{\vee\vee}$ is also free. Since H is torsion free, the canonical map $H \to H^{\vee\vee}$ is injective. $\qquad\square$

COROLLARY 2.5. *Any framed torsion free sheaf (E, Φ) on $\mathbb{P}^2$ is a subsheaf of a locally free sheaf in a canonical way.*

Since E itself is locally free on l_∞, for $r = 1$ the correspondence

$$\begin{array}{ccc} \mathcal{M}^{1,m}(\mathbb{C}^2) & \xrightarrow{\sim} & \mathrm{Hilb}^m(\mathbb{P}^2 \setminus l_\infty) = \mathrm{Hilb}^m(\mathbb{C}^2) \\ E & \mapsto & E^{\vee\vee}/E \end{array}$$

gives an isomorphism.

2.2. Quotient surface singularities and their Hilbert schemes

Let $G < \mathrm{GL}(2, \mathbb{C})$ be a small finite subgroup and denote by $\mathbb{C}^2/G$ the corresponding quotient variety. There are two different types of Hilbert schemes attached to this data. First, there is the classical Hilbert scheme $\mathrm{Hilb}(\mathbb{C}^2/G)$ of the quotient

space. This is the moduli space of ideal sheaves in $\mathcal{O}_{\mathbb{C}^2/G}(\mathbb{C}^2/G)$ of finite colength. We call this the *coarse Hilbert scheme of points*. It decomposes

$$\mathrm{Hilb}(\mathbb{C}^2/G) = \bigsqcup_{m \in \mathbb{N}} \mathrm{Hilb}^m(\mathbb{C}^2/G)$$

into components which are quasiprojective but singular varieties indexed by "the number of points", the codimension m of the ideal. Second, there is the moduli space of G-invariant finite colength subschemes of $\mathbb{C}^2$, the invariant part of $\mathrm{Hilb}(\mathbb{C}^2)$ under the lifted action of G. This Hilbert scheme is also well known and is variously called the *orbifold Hilbert scheme* [64] or *equivariant Hilbert scheme* [25]. We denote it by $\mathrm{Hilb}([\mathbb{C}^2/G])$. This space also decomposes as

$$\mathrm{Hilb}([\mathbb{C}^2/G]) = \bigsqcup_{\rho \in \mathrm{Rep}(G)} \mathrm{Hilb}^\rho([\mathbb{C}^2/G]),$$

where

$$\mathrm{Hilb}^\rho([\mathbb{C}^2/G]) = \{I \in \mathrm{Hilb}(\mathbb{C}^2)^G \colon H^0(\mathcal{O}_{\mathbb{C}^2}/I) \simeq_G \rho\}$$

for any finite-dimensional representation $\rho \in \mathrm{Rep}(G)$ of G; here $\mathrm{Hilb}(\mathbb{C}^2)^G$ is the set of G-invariant ideals of $\mathbb{C}[x, y]$, and $\simeq_G$ means G-equivariant isomorphism. Being components of the fixed point set of a finite group acting on smooth quasiprojective varieties, the orbifold Hilbert schemes themselves are smooth and quasiprojective [7].

There is a natural pushforward map between the two kinds of Hilbert schemes: each $J \in \mathrm{Hilb}([\mathbb{C}^2/G])$ can be mapped to its G-invariant part, giving a morphism [6, 3.4]

$$\begin{aligned} p_* \colon \quad \mathrm{Hilb}([\mathbb{C}^2/G]) \quad &\to \quad \mathrm{Hilb}(\mathbb{C}^2/G) \\ J \quad &\mapsto \quad J^G = J \cap \mathbb{C}[x, y]^G \end{aligned}$$

called the *quotient-scheme map*. There is also a set-theoretic pullback map, which however does *not* preserve flatness in families, so it is not a morphism between the Hilbert schemes: the inclusion $i \colon \mathbb{C}[x, y]^G \subset \mathbb{C}[x, y]$ induces a pullback map on the ideals, and its image is contained in the set of G-equivariant ideals, leading to a map of sets

$$\begin{aligned} i^* \colon \quad \mathrm{Hilb}(\mathbb{C}^2/G)(\mathbb{C}) \quad &\to \quad \mathrm{Hilb}([\mathbb{C}^2/G])(\mathbb{C}) \\ I \quad &\mapsto \quad i^*I = \mathbb{C}[x, y].I \end{aligned}$$

Since for $I \lhd \mathbb{C}[x, y]^G$, we clearly have $(\mathbb{C}[x, y].I)^G = I$, the composite $p_* \circ i^*$ is the identity on the set of ideals of the invariant ring.

We collect the topological Euler characteristics of the two versions of the Hilbert scheme into two generating functions. Let $\rho_0, \ldots, \rho_n \in \mathrm{Rep}(G)$ denote the (isomorphism classes of) irreducible representations of G, with ρ_0 the trivial representation.

DEFINITION 2.6. (a) The *orbifold generating series* of the orbifold $[\mathbb{C}^2/G]$ is

$$Z_{[\mathbb{C}^2/G]}(q_0, \ldots, q_n) = \sum_{m_0, \ldots, m_n = 0}^{\infty} \chi\left(\mathrm{Hilb}^{m_0 \rho_0 + \ldots + m_n \rho_n}([\mathbb{C}^2/G])\right) q_0^{m_0} \cdot \ldots \cdot q_n^{m_n}.$$

(b) The *coarse generating series* of the singularity $\mathbb{C}^2/G$ is just the series defined in (2.1):

$$Z_{\mathbb{C}^2/G}(q) = \sum_{m=0}^{\infty} \chi\left(\mathrm{Hilb}^m(\mathbb{C}^2/G)\right) q^m.$$

Assume that $G < \mathrm{SL}(2,\mathbb{C})$. Then G fixes the line $l_\infty \subset \mathbb{P}^2$ introduced above. Let us take and fix a lift of the G-action to $\mathcal{O}_{l_\infty}^{\oplus r}$. This can be written as $W \otimes_\mathbb{C} \mathcal{O}_{l_\infty}^{\oplus r}$, where W is a representation of G. Then the G-action on $\mathbb{C}^2$ lifts naturally to a G-action on $\mathcal{M}^{r,m}(\mathbb{C}^2)$ [55]. We define the moduli space of G-equivariant torsion free sheaves on $\mathbb{P}^2$ with a framing as the G-invariant part of the space $\mathcal{M}^{r,m}(\mathbb{C}^2)$ with respect to the induced G-action. It will be denoted by $\mathcal{M}^{r,m}([\mathbb{C}^2/G])$. The dependence on the choice of W is suppressed in the notation. In fact, it is easy to see that the isomorphism type of $\mathcal{M}^{r,m}([\mathbb{C}^2/G])$ only depends on the isomorphism class of W. We denote

$$\mathcal{M}^r(\mathbb{C}^2) = \bigsqcup_m \mathcal{M}^{r,m}(\mathbb{C}^2).$$

Using Beilinson's spectral sequence it can be shown that the analog of the quotient $\mathbb{C}[x,y]/I$ for a higher rank framed sheaf (E,ϕ) is $H^1(E(-1))$, and also that $H^0(E(-1)) = H^2(E(-1)) = 0$ [53, Chapter 2]. If E is G-invariant, then $H^1(E(-1))$ carries naturally a G-representation, and there is a decomposition

(2.4) $$\mathcal{M}^r([\mathbb{C}^2/G]) = \bigsqcup_{\rho \in \mathrm{Rep}(G)} \mathcal{M}^{r,\rho}([\mathbb{C}^2/G]),$$

where

$$\mathcal{M}^{r,\rho}([\mathbb{C}^2/G]) = \{(E,\Phi) \in \mathcal{M}^r(\mathbb{C}^2)^G \colon H^1(E(-1)) \simeq_G \rho\}.$$

It is possible to define the higher rank analogue of the coarse Hilbert scheme $\mathrm{Hilb}(\mathbb{C}^2/\mathrm{G})$ and a morphism p_*^G from $\mathcal{M}^r([\mathbb{C}^2/G])$ to this new moduli space which "descends" the sheaves from $[\mathbb{C}^2/G]$ to $\mathbb{C}^2/G$ when W is the trivial r dimensional representation of G. The result of this descent map is a higher rank torsion free sheaf on the variety $\mathbb{P}^2/G$ which is trivial on l_∞/G. The rank one case of this moduli space coincides with $\mathrm{Hilb}(\mathbb{C}^2/\mathrm{G})$. However, the general behaviour and the computations seems to be more complicated in the higher rank case even in type A. Therefore, we leave the investigation of the moduli spaces of higher rank framed torsion free sheaves on quotient singularities for further study.

The Euler characteristics of the higher rank equivariant moduli spaces for a fixed isomorphism class of W are collected again into generating series:

DEFINITION 2.7.

$$Z_{[\mathbb{C}^2/G]}^W(q_0, \ldots, q_n) = \sum_{m_0, \ldots, m_n = 0}^{\infty} \chi\left(\mathcal{M}^{r, m_0\rho_0 + \ldots + m_n\rho_n}([\mathbb{C}^2/G])\right) q_0^{m_0} \cdot \ldots \cdot q_n^{m_n}.$$

In this thesis almost always we are only concerned with finite subgroups $G <$ $\mathrm{SL}(2,\mathbb{C})$. As it is well known, these are classified into three types: type A_n for $n \geq 1$, type D_n for $n \geq 4$ and type E_n for $n = 6, 7, 8$. The type of the singularity can be parameterized by a simply-laced irreducible Dynkin diagram with n nodes, arising from an irreducible simply laced root system Δ. These are the following.

Type A_n: Type D_n:

Type E_6:   Type E_7:

Type E_8:
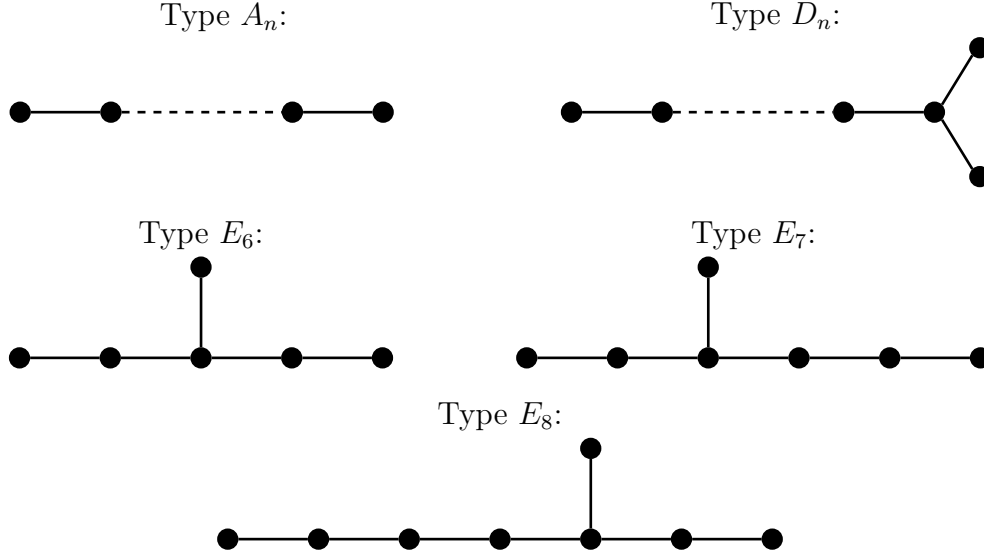
We denote the corresponding group by $G_\Delta < \mathrm{SL}(2,\mathbb{C})$; all other data corresponding to the chosen type will also be labelled by the subscript Δ. Irreducible representations $\rho_0, \ldots, \rho_n$ of G_Δ are then labeled by vertices of the affine Dynkin diagram associated with Δ, which are recalled in 2.3 below. The singularity $\mathbb{C}^2/G_\Delta$ is known as a simple (Kleinian, surface) singularity; we will refer to the corresponding orbifold $[\mathbb{C}^2/G_\Delta]$ as the simple singularity orbifold. Except when noted, we always work with this class of singularities.

Additionally, we will also make some investigation with another class of singularities. Fix a positive integer p. Let $\mathbb{Z}_p$ be the cyclic group of order p with generator g and let it act on $\mathbb{C}^2$ as: $g.x = e^{\frac{2\pi i}{p}}x$, $g.y = e^{\frac{2\pi i q}{p}}y$ where q is coprime to p. Then we get an action of $\mathbb{Z}_p$ on $\mathbb{C}^2$ which is free away from the origin. Let $X(p,q)$ denote the quotient variety. It is called the *cyclic quotient singularity of type* (p,q).

The Hilbert scheme $\mathrm{Hilb}(X(p,q))$ of points on $X(p,q)$ is the moduli space of ideals sheaves in $\mathcal{O}_{X(p,q)}(X(p,q))$ of finite colength. The case when $q = p - 1$ is just the type A singularity introduced above. We will present results about the other extreme case when $q = 1$, that is, about

$$Z_{X(p,1)}(q) = \sum_{m=0}^{\infty} \chi\left(\mathrm{Hilb}^m(X(p,1))\right) q^m.$$

2.3. Quiver variety description of the moduli spaces

Let (I, H) be a quiver. More precisely, I is a set of vertices and H is a set of oriented edges. Let $\overline{H} = H \cup H^*$, where H^* is the set of edges in H with the reversed

orientation. For dimension vectors $\underline{v}, \underline{w} \in \mathbb{Z}_{\geq 0}^I$ we define a *Nakajima quiver variety* as follows. See [51, 55] and references therein for the details, here we follow the notations of [59].

Fix I-graded vector spaces V, W such that $\dim V_i = v_i, \dim W_i = w_i$. Let

$$M(\underline{v}, \underline{w}) = \left(\bigoplus_{h \in \overline{H}} \mathrm{Hom}(V_{\mathrm{s}(h)}, V_{\mathrm{t}(h)}) \right) \oplus \left(\bigoplus_{i \in I} \mathrm{Hom}(W_i, V_i) \oplus \mathrm{Hom}(V_i, W_i) \right),$$

where $h \in \overline{H}$ is an oriented edge from $\mathrm{s}(h)$ to $\mathrm{t}(h)$. Note that $GL(V) = \prod GL(V_i)$ acts on $M(\underline{v}, \underline{w})$ by

$$(g_i) \cdot (B_h, a_i, b_i) = (g_{\mathrm{t}(h)} B_h g_{\mathrm{s}(h)}^{-1}, g_i a_i, b_i g_i^{-1})$$

for any $g_i \in GL(V_i)$, $B_h \in \mathrm{Hom}(V_{\mathrm{s}(h)}, V_{\mathrm{t}(h)})$, $a_i \in \mathrm{Hom}(W_i, V_i)$ and $b_i \in \mathrm{Hom}(V_i, W_i)$. Elements in $M(\underline{v}, \underline{w})$ can be shortly denoted as a triple (B, a, b). Here B is a representation of the path algebra of the quiver on V, while $a \colon W \to V$ and $b \colon V \to W$ are maps of I-graded vector spaces.

The moment map μ for the $GL(V)$-action on $M(\underline{v}, \underline{w})$ is given by

$$\mu(B, a, b) = \bigoplus_{i \in I} \left(\sum_{h : \mathrm{t}(h) = i} \epsilon(h) B_h B_{h^*} + a_i b_i \right) \in \bigoplus_{i \in I} \mathfrak{gl}(V_i) = \mathfrak{gl}(V),$$

where $\epsilon(h) = 1$ and $\epsilon(h^*) = -1$ for $h \in H$. A triple $(B, a, b) \in M(\underline{v}, \underline{w})$ is called *stable* if $\mathrm{im}(a)$ generates V under the action of B. The subset of stable triples in $M(\underline{v}, \underline{w})$ is denoted as $M(\underline{v}, \underline{w})^{\mathrm{st}}$.

The quiver variety associated to the dimension vectors $\underline{v}, \underline{w}$ is

$$\mathcal{M}(\underline{v}, \underline{w}) = \{(B, a, b) \in M(\underline{v}, \underline{w})^{\mathrm{st}} \mid \mu(B, a, b) = 0\} / GL(V).$$

This is well defined only up to a (non-canonical) isomorphism, but since we are only interested in its topological properties we do not consider its dependence on the vector spaces V and W here.

Let us fix an irreducible simply laced root system Δ of rank n. Let (I, H) be the associated *affine* Dynkin quiver. The orientation of the edges in H are specified as in the following diagrams for each type.

Type $\widetilde{A}_n^{(1)}$:

Type $\widetilde{D}_n^{(1)}$:

Type $\widetilde{E}_6^{(1)}$:

Type $\widetilde{E}_7^{(1)}$:

Type $\widetilde{E}_8^{(1)}$:
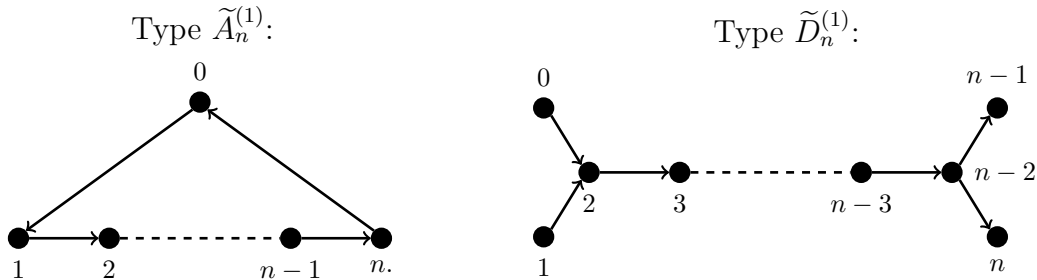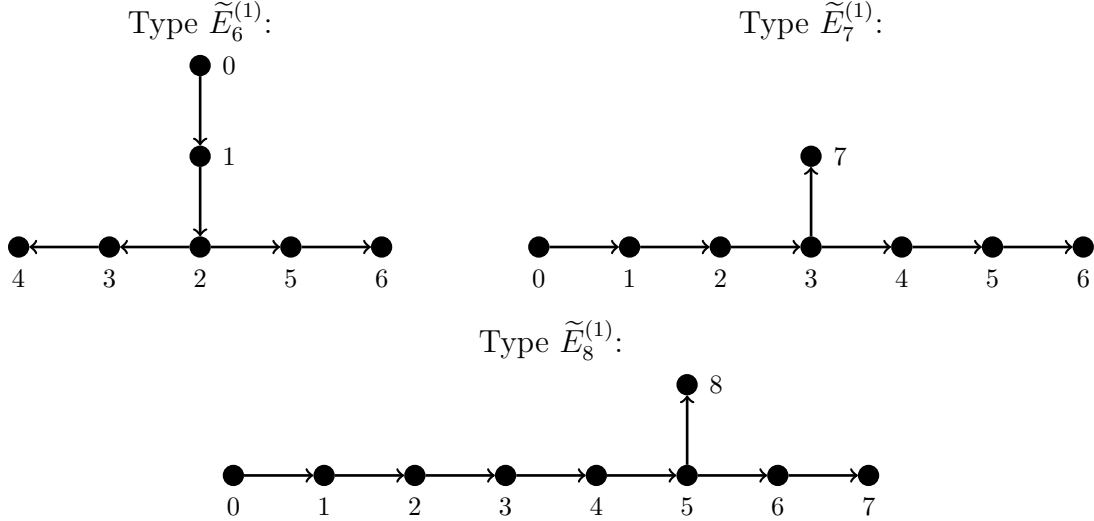
Recall the moduli space $\mathcal{M}^{r,m}([\mathbb{C}^2/G_\Delta])$ of torsion free G_Δ-equivariant sheaves with a framing and its decomposition (2.4). All these depended on the isomorphism class of the G_Δ-representation W, which can be written as $\rho_0^{\oplus w_0} \oplus \cdots \oplus \rho_n^{\oplus w_n}$ where $\rho_0, \ldots, \rho_n$ are the irreducible representations of G. Let $\underline{w} = (w_0, \ldots, w_n)$.

THEOREM 2.8. *[55, Section 3] Let $\rho = \rho_0^{\oplus v_0} \oplus \cdots \oplus \rho_n^{\oplus v_n}$ be the isomorphism class of a G_Δ-representation V, and let $\underline{v} = (v_0, \ldots, v_n)$. The moduli space $\mathcal{M}^{r,\rho}([\mathbb{C}^2/G_\Delta])$ is isomorphic to the quiver variety $\mathcal{M}(\underline{v}, \underline{w})$ associated to (I, H) as above and to the dimension vectors $\underline{v}$ and $\underline{w}$. Therefore, $\mathcal{M}^{r,m}([\mathbb{C}^2/G_\Delta])$ has the decomposition*

$$\mathcal{M}^{r,m}([\mathbb{C}^2/G]) = \bigsqcup_{\underline{v}:|\underline{v}|=m} \mathcal{M}(\underline{v}, \underline{w}).$$

In the case of the Hilbert scheme the representation type of W is necessarily the trivial representation, i.e. $\underline{w} = (1, 0, \ldots, 0)$.

COROLLARY 2.9. *Let $\rho = \rho_0^{\oplus v_0} \oplus \cdots \oplus \rho_n^{\oplus v_n}$ be the isomorphism class of a G_Δ-representation V, and let $\underline{v} = (v_0, \ldots, v_n)$. The Hilbert scheme $\text{Hilb}^\rho([\mathbb{C}^2/G_\Delta])$ is isomorphic to the quiver variety $\mathcal{M}(\underline{v}, \underline{w})$ associated to (I, H) as above and to the dimension vectors $\underline{v}$ and $\underline{w} = (1, 0, \ldots, 0)$. Therefore, $\text{Hilb}^m([\mathbb{C}^2/G_\Delta])$ has the decomposition*

$$\text{Hilb}^m([\mathbb{C}^2/G_\Delta]) = \bigsqcup_{\underline{v}:|\underline{v}|=m} \mathcal{M}(\underline{v}, \underline{w}).$$

2.4. Representations of affine Lie algebras

The author learned the material in this section from Balázs Szendrői.

Let Δ be an irreducible finite-dimensional root system, corresponding to a complex finite dimensional simple Lie algebra $\mathfrak{g}$ of rank n. Attached to Δ is also an (untwisted) affine Lie algebra $\tilde{\mathfrak{g}}$, but a slight variant will be more interesting for us, see e.g. [15, Sect 6]. Consider the Lie algebra $\widetilde{\mathfrak{g} \oplus \mathbb{C}}$ spanned by elements xt^m for $x \in \mathfrak{g}$, $m \in \mathbb{Z}$; elements a_j for $j \in \mathbb{Z} \setminus \{0\}$, a scaling element d, and a central element c. It is the

direct sum of the affine Lie algebra $\tilde{\mathfrak{g}}$ and an infinite Heisenberg algebra $\mathfrak{heis}$, with their centers identified.

Let V_0 be the basic representation of $\tilde{\mathfrak{g}}$, the level-1 representation with highest weight ω_0. Let $\mathcal{F}$ be the standard Fock space representation of $\mathfrak{heis}$, having central charge 1. Then $V = V_0 \otimes \mathcal{F}$ is a representation of $\widetilde{\mathfrak{g} \oplus \mathbb{C}}$ that we may call the extended basic representation. By the Frenkel–Kac theorem [17],

$$V \cong \mathcal{F}^{n+1} \otimes \mathbb{C}[Q_\Delta],$$

where Q_Δ is the root lattice corresponding to the root system Δ. Here, for $\beta \in \mathbb{C}[Q_\Delta]$, $\mathcal{F}^{n+1} \otimes e^\beta$ is the sum of weight subspaces of weight $\omega_0 - \left(m + \frac{\langle \beta, \beta \rangle}{2} \right) \delta + \beta$, $m \geq 0$, with δ being the imaginary root [36]. Thus, we can write the character of this representation as

$$(2.5) \quad \mathrm{char}_V(q_0, \ldots, q_n) = e^{\omega_0} \left(\prod_{m>0} (1 - q^m)^{-1} \right)^{n+1} \cdot \sum_{\beta \in Q_\Delta} q_1^{\beta_1} \cdot \cdots \cdot q_n^{\beta_n} (q^{1/2})^{\langle \beta, \beta \rangle},$$

where $q = e^{-\delta}$, and $\beta = (\beta_1, \ldots, \beta_n)$ is the expression of an element of the finite type root lattice in terms of the simple roots.

EXAMPLE 2.10. For Δ of type A_n, we have $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $\tilde{\mathfrak{g}} = \widetilde{\mathfrak{sl}}_{n+1}$, $\widetilde{\mathfrak{g} \oplus \mathbb{C}} = \widetilde{\mathfrak{gl}}_{n+1}$. In this case there is in fact a natural vector space isomorphism $V \cong \mathcal{F}$ with Fock space itself, see e.g. [61, Section 3E].

At least for simply-laced classical types A_n and D_n (and also for some others, see [39]), the above representation can be constructed on a vector space spanned by an explicit (affine) "crystal" basis. In type A, this construction is very well known [49]; the type D construction is more recent [37, 38]. For completeness, we recall the definition from [41].

Let $\tilde{\mathfrak{g}}$ be an affine Kac-Moody algebra, $\{h_i\}_{i \in I}$ be the set of simple coroots, $\{\alpha_i\}_{i \in I}$ the set of simple roots, and P the weight lattice.

DEFINITION 2.11. An *(affine) crystal basis* for $\tilde{\mathfrak{g}}$ is a set B together with maps $\mathrm{wt} \colon B \to P$, $\varepsilon_i, \varphi_i \colon B \to \mathbb{Z} \cup \{-\infty\}$, and $\tilde{e}_i, \tilde{f}_i \colon B \to B \cup \{0\}$ for $i \in I$, satisfying the following conditions; for $i \in I$, and $b, b' \in B$

    (1) $\varphi_i(b) = \varepsilon_i(b) + \mathrm{wt}(b)(h_i)$,
    (2) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$, and $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$, if $\tilde{e}_i b \in B$,
    (3) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$, and $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) - \alpha_i$, if $\tilde{e}_i b \in B$,
    (4) $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$,
    (5) if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

In [32] for many types of affine Lie algebras crystal bases are constructed on certain combinatorial objects called *proper Young walls*. In type A and D we will describe these in detail in the later chapters. For Δ of type A or D the set of

proper Young walls will be denoted as $\mathcal{Z}_\Delta$. To be more precise, the Young wall pattern for A_n, (respectively, D_n) introduced in Chapter 4 (respectively, in Chapter 5) corresponds to the dominant integral weight $\Lambda = \Lambda_0$ representation of the affine Lie algebra $\mathfrak{g}$ of type $\tilde{A}_n$ (resp., $\tilde{D}_n$).

In both type A and D there is a certain subset $\mathcal{Y}_\Delta \subset \mathcal{Z}_\Delta$ of Young walls which are called *reduced*. These satisfy further combinatorial properties which we omit here, see [38] for the precise definitions. It is shown in [43] for type A_n, and in [41] for type D_n, that there is a bijection

$$\mathcal{Y}_\Delta \longleftrightarrow \mathcal{C}_\Delta \times \mathcal{P}^n \ .$$

Here $\mathcal{C}_\Delta$ is the set of core Young wall as in the main part of the text.

By [49, 37] the basic representation V_0 of $\tilde{\mathfrak{g}}$ can be constructed on $\mathcal{Y}_\Delta$, and it is canonically embedded into the extented basic representation of $\widetilde{\mathfrak{g} \oplus \mathbb{C}}$ constructed on $\mathcal{Z}_\Delta$. The embedding is induced by the inclusion $\mathcal{Y}_\Delta \subset \mathcal{Z}_\Delta$. The character of the basic representation is given by the Weyl-Kac character formula

$$\mathrm{char}_{V_0}(q_0, \ldots, q_n) = e^{\omega_0} \left(\prod_{m>0} (1 - q^m)^{-1} \right)^n \cdot \sum_{\beta \in Q_\Delta} q_1^{\beta_1} \cdot \cdots \cdot q_n^{\beta_n} (q^{1/2})^{\langle \beta, \beta \rangle} \ .$$

As before, let $\Gamma < \mathrm{SL}(2, \mathbb{C})$ be a finite subgroup and let $\Delta \subset \tilde{\Delta}$ be the corresponding finite and affine Dynkin diagrams. By Corollary 2.9 the equivariant Hilbert schemes $\mathrm{Hilb}^\rho([\mathbb{C}^2/\Gamma])$ for all finite dimensional representations ρ of G_Γ are Nakajima quiver varieties [51] associated to $\tilde{\Delta}$, with dimension vector determined by ρ, and a specific stability condition. Ssee [18, 50] for more details for type A.

Nakajima's general results on the relation between the cohomology of quiver varities and Kac-Moody algebras, specialized to this case, imply

THEOREM 2.12 ([51, 55]). *The direct sum of all cohomology groups* $H^*(\mathrm{Hilb}^\rho([\mathbb{C}^2/G]))$ *is graded isomorphic to the extended basic representation V of the corresponding extended affine Lie algebra $\widetilde{\mathfrak{g} \oplus \mathbb{C}}$ defined above.*

CHAPTER 3

The main results and their corollaries

This chapter announces the main results of the thesis. As a corollary, a modularity result for the partition function $Z_X(q)$ is proved for certain singular surfaces X, extending the S-duality type result from the nonsingular case.

3.1. The main results

By 2.3 the Hilbert schemes on $[\mathbb{C}^2/G_\Delta]$ where Δ is an irreducible simply-laced root system have a description as Nakajima quiver varieties. By [54, Section 7], these quiver varieties have no odd cohomology. Taking into account Theorem 2.12, the character formula (2.5) implies the following theorem.

THEOREM 3.1 ([55]). *Let $[\mathbb{C}^2/G_\Delta]$ be a simple singularity orbifold, where Δ is any of the types A, D, and E. Then its orbifold generating series can be expressed as*

(3.1)
$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_n) = \left(\prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{n+1} \cdot \sum_{\overline{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n} q_1^{m_1} \cdots q_n^{m_n} (q^{1/2})^{\overline{m}^\top \cdot C_\Delta \cdot \overline{m}},$$

where $q = \prod_{i=0}^{n} q_i^{d_i}$ with $d_i = \dim \rho_i$, and C_Δ is the finite type Cartan matrix corresponding to Δ.

Our first main result is a strengthening of this theorem. Given a Dynkin diagram Δ of type A or D, we will recall below in 4.2, respectively 5.2, the definition of a certain combinatorial set, the *set of Young walls $\mathcal{Z}_\Delta$ of type Δ*.

THEOREM 3.2. *Let $[\mathbb{C}^2/G_\Delta]$ be a simple singularity orbifold, where Δ is of type A_n for $n \geq 1$ or D_n for $n \geq 4$. Then there exists a decomposition*

$$\mathrm{Hilb}([\mathbb{C}^2/G_\Delta]) = \bigsqcup_{Y \in \mathcal{Z}_\Delta} \mathrm{Hilb}([\mathbb{C}^2/G_\Delta])_Y$$

into locally closed strata indexed by the set of Young walls $\mathcal{Z}_\Delta$ of the appropriate type. Each stratum is isomorphic to an affine space of a certain dimension, and in particular has Euler characteristic $\chi(\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])_Y) = 1$.

For type A, the set of Young walls is simply the set of finite partitions, represented as Young diagrams, equipped with a diagonal labelling. In this case, Theorem 3.2 is well known; the decomposition in type A is not unique, but depends on a choice of a one-dimensional subtorus of the full torus $(\mathbb{C}^*)^2$ acting on the affine plane $\mathbb{C}^2$. For completeness, we summarize the details in Chapter 4. On the other hand, the

type D case appears to be new; in this case, our decomposition is unique, there is no further choice to make.

REMARK 3.3. As it was explained in 2.3, the orbifold Hilbert schemes of points for $G < \mathrm{SL}(2, \mathbb{C})$ are Nakajima quiver varieties for the corresponding affine quiver. As it was shown in [60], certain Lagrangian subvarieties in Nakajima quiver varieties are isomorphic to quiver Grassmannians for the preprojective algebra of the same type, parameterizing submodules of certain fixed modules. On the other hand, results of the recent papers [45, 46] imply that every quiver Grassmannian of a representation of a quiver of affine type D has a decomposition into affine spaces. The relation between this decomposition and ours deserves further investigation.

We will explain combinatorially in 4.4, respectively 9.3, that the right hand side of (3.1) enumerates the set of Young walls $\mathcal{Z}_\Delta$ of the appropriate type. The same combinatorics appears in the representation theoretic considerations in Section 2.4. Thus Theorem 3.2 implies Theorem 3.1.

REMARK 3.4. In type A, it is easy to refine formula (3.1) to a formula involving the Betti numbers [18], or the motives [25], of the orbifold Hilbert schemes. We leave the study of such a refinement in type D to future work; compare to Remark 6.4.

The second main result of the thesis is the following formula, which says that the coarse generating series is a very particular specialization of the orbifold one.

THEOREM 3.5. *Let $\mathbb{C}^2/G_\Delta$ be a simple singularity, where Δ is of type A_n for $n \geq 1$ or D_n for $n \geq 4$. Let $h^\vee$ be the (dual) Coxeter number of the corresponding finite root system (one less than the dimension of the corresponding simple Lie algebra divided by n). Then*

$$Z_{\mathbb{C}^2/G_\Delta}(q) = \left(\prod_{m=1}^\infty (1 - q^m)^{-1} \right)^{n+1} \cdot \sum_{\overline{m}=(m_1,\ldots,m_n) \in \mathbb{Z}^n} \zeta^{m_1+m_2+\cdots+m_n} (q^{1/2})^{\overline{m}^\top \cdot C_\Delta \cdot \overline{m}},$$

where $\zeta = \exp\left(\frac{2\pi i}{1+h^\vee}\right)$ and C_Δ is the finite type Cartan matrix corresponding to Δ.

Thus $Z_{\mathbb{C}^2/G_\Delta}(q)$ is obtained from $Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_n)$ by the substitutions

$$q_1 = \cdots = q_n = \exp\left(\frac{2\pi i}{1+h^\vee}\right), \quad q_0 = q \exp\left(-\frac{2\pi i}{1+h^\vee} \sum_{i \neq 0} \dim \rho_i\right).$$

REMARK 3.6. The single variable generating series $Z_{\mathbb{C}^2/G_\Delta}$ in type A was calculated by Toda in [62] using threefold machinery including a flop formula for Donaldson–Thomas invariants of certain Calabi–Yau threefolds. He does not mention any connection to Lie theory. The combinatorics, and the one-variable formula for $Z_\Delta^0(q)$, were already known to Dijkgraaf and Sułkowski [12]. They do not give the interpretation of the combinatorial formula in terms of Hilbert schemes, though

they are clearly motivated by closely related ideas. Instead, they investigate the partition functions of certain supersymmetric field theories on the so-called ALE spaces (see [11, 9] for further developments in this direction). The method in [12] for the proof of Theorem 3.5 in type A is different, using the method of Andrews [2]. This has motivated a new proof of us for Theorem 3.2 in type A appearing in [26], which we explain here in Section 4.3. Our other main contribution is the general Lie-theoretic formulation, as well as a proof in type D described in Chapters 5-9 as in [28]. Along the way, we also provide another direct combinatorial proof in type A as well, which appears to be new. We believe that already in type A our new proof is preferable since it directly exhibits the clear connection between the orbifold and coarse generating series. Also, as we show in the second part of the thesis, this method generalizes away from type A.

One can check directly that the generating series in Theorem 3.5 has also integer coefficients for E_6, E_7 and E_8 to a high power in q. This motivates the following.

CONJECTURE 3.7. *Let $\mathbb{C}^2/G_\Delta$ be a simple singularity of type E_n for $n = 6, 7, 8$. Let $h^\vee$ be the (dual) Coxeter number of the corresponding finite root system. Then, as for other types,*

$$Z_{\mathbb{C}^2/G_\Delta}(q) = \left(\prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{n+1} \cdot \sum_{\overline{m}=(m_1,\ldots,m_n)\in\mathbb{Z}^n} \zeta^{m_1+m_2+\cdots+m_n}(q^{1/2})^{\overline{m}^\top \cdot C_\Delta \cdot \overline{m}},$$

where $\zeta = \exp\left(\frac{2\pi i}{1+h^\vee}\right)$ and C_Δ is the finite type Cartan matrix corresponding to Δ.

The key tool in our proof of Theorem 3.5 for types A and D is the combinatorics of Young walls, in particular their abacus representation. We are not aware of such explicit combinatorics in type E. We hope to return to this question in later work.

REMARK 3.8. We are dealing here with Hilbert schemes, parameterizing rank $r = 1$ sheaves on the orbifold or singular surface. In the relationship between the instantons on algebraic surfaces and affine Lie algebras, level equals rank [23]. Indeed the (extended) basic represenatation underlying the Young wall combinatorics (see Appendix) has level $l = 1$. Thus the substitution above is by the root of unity $\zeta = \exp\left(\frac{2\pi i}{l+h^\vee}\right)$, with $l = 1$ and $h^\vee$ the (dual) Coxeter number. There is an intriguing analogy here with the Verlinde formula, which uses a similar substitution, into characters of Lie algebras, by a root of unity $\zeta = \exp(\frac{2\pi i}{l+h^\vee})$, where l again is the level, and $h^\vee$ the (dual) Coxeter number of the root system of the Lie algebra of the gauge group. The geometric significance of this observation, if any, is left for future research.

In the type A case we can go further to the higher rank case. The higher rank orbifold generating series can again be derived from the results of [55] or [18], but our methods give a new proof in this case as well. Let W be the isomorphism class

of the framing and let $\underline{a} = (0, \ldots, 0, \ldots, n-1, \ldots, n-1)$, where for each $c \in C$ the number of c's in $\underline{a}$ is w_c.

THEOREM 3.9. *Let* $[\mathbb{C}^2/G_\Delta]$ *be a simple singularity orbifold, where* Δ *is of type* A_n *for* $n \geq 1$. *Then*

$$Z^W_{[\mathbb{C}^2/G_\Delta]}(\underline{q}) = \prod_{m=1}^{l} Z_{\Delta(a_m)}(\underline{q}),$$

where l *is the length of* $\underline{a}$,

$$Z_{\Delta(a)}(\underline{q}) = \left(\prod_{m=1}^{\infty} (1-q^m)^{-1}\right)^{n+1} \cdot \sum_{\underline{m}=(m_1,\ldots,m_n)\in\mathbb{Z}^n} q_{1+a}^{m_1} \cdot \cdots \cdot q_{n+a}^{m_n} (q^{1/2})^{\underline{m}^\top \cdot C \cdot \underline{m}},$$

$q = \prod_{i=0}^{n} q_i$, *and* C *is the Cartan matrix of finite type* A_n.

Let us consider now the cyclic quotient singularities introduced in the end of Section 2.2. Our main result in this direction is a representation of $Z_{X(p,1)}(q)$ as coefficient of a two variable generating function. In this two variable generating function continued fractions appear. We introduce the notation $[z^0] \sum_n A_n z^n = A_0$.

THEOREM 3.10. *Let* $F(q,z)$ *and* $T(q,z)$ *be the functions defined in* (4.17) *and* (4.20) *below, respectively. Then:*

$$Z_{X(p,1)}(q) = [z^0]T(q,z)\left(F(q^{-1},z^{-1}) - (qz)^{-p}F(q^{-1},(qz)^{-1})\right).$$

REMARK 3.11.   (1) For $p = 1$, $X(1,1) = \mathbb{C}^2$. Then, by Theorem 3.5, $Z_{X(1,1)}(q)$ is also equal to

$$\prod_{m=1}^{\infty} \frac{1}{1-q^m}.$$

(2) For $p = 2$, $X(2,1)$ is the A_1 singularity. By Theorem 3.5, $Z_{X(2,1)}(q)$ is also equal to

$$\left(\prod_{m=1}^{\infty}(1-q^m)^{-1}\right)^2 \cdot \sum_{m\in\mathbb{Z}} \xi^m q^{m^2},$$

where $\xi = \exp(\frac{2\pi i}{3})$.
We are not aware of a direct proof for these equalities.

Finally, we are able to globalize the statements so far. Let $X = S$ be a quasi-projective surface which is non-singular outside a finite number of simple surface singularities $\{P_1, \ldots, P_k\}$, with $(P_i \in S)$ a singularity locally analytically isomorphic to $(0 \in \mathbb{C}^2/G_{\Delta_i})$ for $G_{\Delta_i} < \mathrm{SL}(2,\mathbb{C})$ a small finite subgroup, or to $(0 \in (X(p_i,1)))$ for a positive integer p_i. Let $S^0 \subset S$ be the nonsingular part of S.

THEOREM 3.12. *The generating function $Z_S(q)$ of the Euler characteristics of Hilbert schemes of points of S has a product decomposition*

(3.2)
$$Z_S(q) = \left(\prod_{m=1}^{\infty} (1-q^m)^{-1} \right)^{\chi(S^0)} \cdot \prod_{j=1}^{k} Z^{(P_i,S)}(q).$$

The local terms can be expressed

- *either as*

(3.3)
$$Z^{(P_i,S)}(q) = Z_{\mathbb{C}^2/G_{\Delta_i}}(q)$$

 if $(P_i \in S) \cong (0 \in \mathbb{C}^2/G_{\Delta_i})$, and they are given by Theorem 3.5 for type A and D, and, assuming Conjecture 3.7, also for type E;

- *or as*

(3.4)
$$Z^{(P_i,S)}(q) = Z_{X(p_i,1)}(q)$$

 if $(P_i \in S) \cong (0 \in (X(p_i,1)))$, and they are given by Theorem 3.10.

PROOF. The product decomposition (3.2), as well as the equalities (3.3)-(3.4), follow from a standard argument; we sketch the details. For a point $P \in S$ on a quasiprojective surface S, let $\mathrm{Hilb}_P^m(S)$ denote the punctual Hilbert scheme of S at P, the Hilbert scheme of length m subschemes of S set-theoretically supported at the single point P. Then for our surface S and for a, let's say, simple singularity P_i we have

$$\chi(\mathrm{Hilb}_{P_i}^m(S)) = \chi(\mathrm{Hilb}_0^m(\mathbb{C}^2/G_{\Delta_i})) = \chi(\mathrm{Hilb}^m(\mathbb{C}^2/G_{\Delta_i})).$$

Here the first equality follows from the analytic isomorphism between $(P_i \in S)$ and $(0 \in \mathbb{C}^2/G_{\Delta_i})$. The second equality follows from torus localization, using the fact that each singularity $\mathbb{C}^2/G_{\Delta_i}$ is weighted homogeneous, admitting a $\mathbb{C}^*$-action. Any finite length quotient whose set-theoretic support is not equal to 0 is a member of a nontrivial orbit of this action. Therefore,

$$\chi(\mathrm{Hilb}^m(\mathbb{C}^2/G_{\Delta_i}) \setminus \mathrm{Hilb}_0^m(\mathbb{C}^2/G_{\Delta_i})) = 0,$$

and we have the decomposition

$$\mathrm{Hilb}^m(S) = \bigsqcup_{\sum_{i=0}^{k} m_i = m} \mathrm{Hilb}^{m_0}(S^0) \times \prod_{i=1}^{k} \mathrm{Hilb}_{P_i}^{m_i}(S).$$

Reinterpreting this equality for generating series proves (3.2)-(3.4), recalling also (2.2) for S^0. $\square$

Formulas (3.2)-(3.3) are our analogue for the case of surfaces with simple singularities of the Oblomkov–Shende–Maulik formula (2.3). Note that each $\mathbb{C}^2/G_{\Delta_i}$ is in particular a hypersurface singularity, as are planar singularities in the curve case. The main difference with formula (2.3) is the fact that (conjecturally, for type E) our local terms $Z^{(P_i,S)}(q)$ are expressed in terms of Lie-theoretic and not topological

data. We leave the question whether our local terms have any interpretation in terms of the topology of the (embedded) link of P_i, and whether there are nice formulas for other two-dimensional hypersurface singularities, for further work.

3.2. The S-duality conjecture

We follow here [22]. Let S be a smooth projective algebraic surface and let $H = \mathcal{O}(1)$ be a very ample line bundle on S. For a sheaf E on S, denote $\chi(S, E) = \sum_{i=0}^{2}(-1)^i \dim H^i(S, E)$ and let $E(n) = E \otimes H^{\otimes n}$. Let $r = \mathrm{rk}(E)$ be the rank of E. The *discriminant* of a coherent sheaf E is by definition [33, Section 3.4] the characteristic class

$$\Delta(E) = c_2 - \frac{r-1}{2r}c_1^2.$$

We do not distinguish between $\Delta(E)$ and its degree

$$d = \int_X \Delta(E).$$

DEFINITION 3.13. A torsion free sheaf E on S is *H-semistable*, if for all nonzero subsheaves $F \subset E$, we have

$$\frac{\chi(S, F(n))}{\mathrm{rk}(F(n))} \leq \frac{\chi(S, E(n))}{\mathrm{rk}(E(n))}, \quad \text{for all } n \gg 0.$$

It is called *H-stable* if the inequality is strict.

It can be shown that there exists a moduli space $\mathcal{M}_S(r, c_1, c_2)$ of H-semistable sheaves on S of rank r with Chern classes c_1 and c_2. The Hilbert scheme $\mathrm{Hilb}^m(S)$ of S is the special case with rank 1, $c_1 = 0$ and $c_2 = m$. For any torsion free and H-semistable sheaf E Bogomolov's inequality [33, Theorem 3.4.1] states that

$$d \geq 0.$$

For H-stable sheaves this inequality is strict.

Let us restrict our attention to the case of rank 2. Then

$$d = c_2 - \frac{c_1^2}{4},$$

and we denote the moduli spaces as $\mathcal{M}_S(c_1, d) = \mathcal{M}_S(2, c_1, c_2)$. The Euler characteristics are again collected into a generating series

$$Z_{S,c_1}(q) = \sum_d \chi(\mathcal{M}_S(c_1, d))q^d.$$

Vafa and Witten interpret this generating function in [63] as the partition function of a physical theory. The variable q is interpreted as $q = \mathrm{e}^{2\pi i \tau}$, where τ is a coupling parameter of the theory. To write it in this form, it has to be invariant under the transformation $T \colon \tau \to \tau + 1$. Furthermore, the theory should transform nicely if one replaces the strong coupling with weak coupling. This corresponds to the transformation $S \colon \tau \to -\frac{1}{\tau}$. The operations S and T generate the action of $SL(2, \mathbb{Z})$

on the upper half plane $\mathbb{H}$. The *S-duality conjecture* desribes how the theory and associated quantities transform with respect to the $SL(2, \mathbb{Z})$-action. In particular, it predicts that, up to a fractional power of q, $Z_{S,c_1}(q)$ has to be a meromorphic modular form for a finite index subgroup of $SL(2, \mathbb{Z})$.

It was observed by Balázs Szendrői that our formulae from Section 3.1 lead to the following new modularity results, extending the results of [62] for type A.

COROLLARY 3.14. *(S-duality for simple singularities) For type A and type D, and, assuming Conjecture 3.7, for all types, the partition function $Z_{\mathbb{C}^2/G_\Delta}(q)$ is, up to a suitable fractional power of q, the q-expansion of a meromorphic modular form of weight $-\frac{1}{2}$ for some congruence subgroup of* $\mathrm{SL}(2, \mathbb{Z})$.

PROOF. This follows straight from [62, Prop.3.2]. $\qquad\square$

COROLLARY 3.15. *(S-duality for surfaces with simple singularities) Let S be a quasiprojective surface with simple singularities of type A and D, or, assuming Conjecture 3.7, of arbitrary type. Then the generating function $Z_S(q)$ is, up to a suitable fractional power of q, the q-expansion of a meromorphic modular form of weight $-\frac{\chi(S)}{2}$ for some congruence subgroup of* $\mathrm{SL}(2, \mathbb{Z})$.

PROOF. Combine Theorem 3.12 and Corollary 3.14. $\qquad\square$

Abelian quotient singularities

This section gives all the proofs for the main results in the type A case. First, the geometric problem is reduced to partition enumeration. Two independent solutions to the enumerative problem are given. Finally a closely related case of cyclic quotient singularities is treated.

4.1. Type A basics

Let Δ be the root system of type A_n. Choosing a primitive $(n+1)$-st root of unity ω, the corresponding subgroup G_Δ of $SL(2,\mathbb{C})$, a cyclic subgroup of order $n+1$, is generated by the matrix

$$\sigma = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}.$$

All irreducible representations of G_Δ are one dimensional, and they are simply given by $\rho_j \colon \sigma \mapsto \omega^j$, for $j \in \{0, \ldots, n\}$. The corresponding McKay quiver is the cyclic Dynkin diagram of type $\widetilde{A}_n^{(1)}$.

The group G_Δ acts on $\mathbb{C}^2$; the quotient variety $\mathbb{C}^2/G_\Delta$ has an A_n singularity at the origin. The matrix σ clearly commutes with the diagonal two-torus $T = (\mathbb{C}^*)^2$, and so T acts on the quotient $\mathbb{C}^2/G_\Delta$ and the orbifold $[\mathbb{C}^2/G_\Delta]$. Consequently T also acts on the orbifold Hilbert scheme $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])$ and the (reduced) coarse Hilbert scheme $\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)$ as well.

4.2. Partitions, torus-fixed points and decompositions

Consider the set $\mathbb{N} \times \mathbb{N}$ of pairs of non-negative integers; we will draw this set as a set of blocks on the plane, occupying the non-negative quadrant. Label blocks diagonally with $(n+1)$ labels $0, \ldots, n$ as in the picture; the block with coordinates (i,j) is labelled with $(i-j) \mod (n+1)$. We will call this the *pattern of type A_n*, or the *diagonal labelling*.

⋮							
0	1						
1	2						
⋮	⋮						
n	0			$n-2$	$n-1$	n	0
0	1	$\cdots$	$n-1$	n	0	1	$\cdots$

Let $\mathcal{P}$ denote the set of partitions. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}$, with $\lambda_1 \geq \ldots \geq \lambda_k$ positive integers, we consider its Young (or Ferrers) diagram, the subset of $\mathbb{N} \times \mathbb{N}$ which consists of the λ_i lowest blocks in column $i-1$. The blocks in λ also get labelled (or colored) by the $n+1$ labels. Let $\mathcal{Z}_\Delta$ denote the resulting set of diagonally labelled partitions, including the empty partition. For a labelled partition $\lambda \in \mathcal{Z}_\Delta$, let $\mathrm{wt}_j(\lambda)$ denote the number of blocks in λ labelled j, and define the multiweight of λ to be $\underline{\mathrm{wt}}(\lambda) = (\mathrm{wt}_0(\lambda), \ldots, \mathrm{wt}_n(\lambda))$.

PROPOSITION 4.1. *The torus T acts with isolated fixed points on* $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])$, *parametrized by the set $\mathcal{Z}_\Delta$ of diagonally labelled partitions. More precisely, for $k_0, \ldots, k_n$ non-negative integers and $\rho = \oplus_{j=0}^n \rho_i^{\oplus k_i}$, the T-fixed points on* $\mathrm{Hilb}^\rho([\mathbb{C}^2/G_\Delta])$ *are parametrized by diagonally labelled partitions of multiweight* $(k_0, \ldots, k_n)$.

PROOF. We just sketch the proof, which is well known [14, 18]. It is clear that the T-fixed points on $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])$, which coincide with the T-fixed points on $\mathrm{Hilb}(\mathbb{C}^2)$, are the monomial ideals in $\mathbb{C}[x, y]$ of finite colength. The monomial ideals are enumerated in turn by Young diagrams of partitions. The labelling of each block gives the weight of the G_Δ-action on the corresponding monomial, proving the refined statement. $\square$

COROLLARY 4.2. *There exist a locally closed decomposition, depending on a choice specified below, of* $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])$ *into strata indexed by the set of diagonally labelled partitions. Each stratum is isomorphic to an affine space.*

PROOF. Again, this is well known [56]. Fixing a representation ρ, choose a sufficiently general one-dimensional subtorus $T_0 \subset T$ which has positive weight on both x and y. For general $T_0 \subset T$, the fixed point set on $\mathrm{Hilb}^\rho([\mathbb{C}^2/G_\Delta])$ is unchanged and in particular consists of a finite number of isolated points. Choosing positive weights on x, y ensures that all limits of T_0-orbits at $t = 0$ in $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])$ exist, even though $\mathrm{Hilb}^\rho([\mathbb{C}^2/G_\Delta])$ is non-compact. Since $\mathrm{Hilb}^\rho([\mathbb{C}^2/G_\Delta])$ is smooth, the result follows by taking the Bialynicki-Birula decomposition of $\mathrm{Hilb}^\rho([\mathbb{C}^2/G_\Delta])$ given by the T_0-action. $\square$

Denote by

$$Z_\Delta(q_0, \ldots, q_n) = \sum_{\lambda \in \mathcal{Z}_\Delta} \underline{q}^{\mathrm{wt}(\lambda)}$$

the generating series of diagonally labelled partitions, where we used multi-index notation

$$\underline{q}^{\mathrm{wt}(\lambda)} = \prod_{i=0}^{n} q_i^{\mathrm{wt}_i(\lambda)}.$$

From either of the previous two statements, we immediately deduce the following.

COROLLARY 4.3. *Let $[\mathbb{C}^2/G_\Delta]$ be a simple singularity orbifold of type A. Then its orbifold generating series can be expressed as*

(4.1) $$Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_n) = Z_\Delta(q_0, \ldots, q_n).$$

PROPOSITION 4.4. *The generating series of diagonally labelled partitions has the following form:*

(4.2) $$Z_\Delta(q_0, \ldots, q_n) = \frac{\sum_{\overline{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^k}^{\infty} q_1^{m_1} \cdot \ldots \cdot q_n^{m_n} (q^{1/2})^{\overline{m}^\top \cdot C \cdot \overline{m}}}{\prod_{m=1}^{\infty} (1 - q^m)^{n+1}},$$

where $q = q_0 \cdot \ldots \cdot q_n$ and C is the (finite) Cartan matrix of type A_n. In particular, (4.1) and (4.2) imply Theorem 3.1 for type A.

From the several existing proofs of this theorem we will give two below. The first one, explained in 4.3, is an easy and direct combinatorial argument and was found by the author [26]. The second one, which is summarized in 4.4, has originally appeared in [18]. It has the advantage that is gives the starting point for the analysis in type D.

We now turn to the coarse Hilbert scheme. Let us define a subset $\mathcal{Z}_\Delta^0$ of the set of diagonally labelled partitions $\mathcal{Z}_\Delta$ as follows. An diagonally labelled partition $\lambda \in \mathcal{Z}_\Delta$ will be called 0-*generated* (a slight misnomer, this should be really be "complement-0-generated") if the complement of λ inside $\mathbb{N} \times \mathbb{N}$ can be completely covered by translates of $\mathbb{N} \times \mathbb{N}$ to blocks labelled 0 contained in this complement. Equivalently, an diagonally labelled partition λ is 0-generated, if all its addable blocks (blocks whose addition gives another partition) are labelled 0. It is immediately seen that this condition is equivalent to the corresponding monomial ideal $I \lhd \mathbb{C}[x, y]$ being generated by its invariant part $I \cap \mathbb{C}[x, y]^{G_\Delta}$. Indeed, we have the following.

PROPOSITION 4.5. *The torus T acts with isolated fixed points on $\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)$, which are in bijection with the set $\mathcal{Z}_\Delta^0$ of 0-generated diagonally labelled partitions. More precisely, for a non-negative integer k, the T-fixed points on $\mathrm{Hilb}^k(\mathbb{C}^2/G_\Delta)$ are parametrized by 0-generated diagonally labelled partitions λ with 0-weight $\mathrm{wt}_0(\lambda) = k$.*

PROOF. This is immediate from the above discussion. The T-fixed points of $\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)$ are the monomial ideals I of $\mathbb{C}[x, y]^{G_\Delta}$ of finite colength. Inside $\mathbb{C}[x, y]$,

the ideals they generate correspond to partitions which are 0-generated. The ring $\mathbb{C}[x,y]^{G_\Delta}$ has a basis consisting of monomials with corresponding blocks labelled 0 inside $\mathbb{C}[x,y]$; thus the codimension of a monomial ideal I inside $\mathbb{C}[x,y]^{G_\Delta}$ is simply the number of blocks denoted 0. $\square$

Denoting by

$$Z^0_\Delta(q) = \sum_{\lambda \in \mathcal{Z}^0_\Delta} q^{\mathrm{wt}_0(\lambda)}$$

the corresponding specialization of the generating series of 0-generated diagonally labelled partitions, we deduce the following.

COROLLARY 4.6. *Let $[\mathbb{C}^2/G_\Delta]$ be a simple singularity orbifold of type A. Then the coarse generating series can be expressed as*

(4.3) $$Z_{\mathbb{C}^2/G_\Delta}(q) = Z^0_\Delta(q).$$

PROOF OF THEOREM 3.5 FOR THE A_n CASE. The (dual) Coxeter number of the type A_n root system is $h^\vee = n+1$. Thus Theorem 3.5 for this case follows from Corollary 4.6, formula (4.2), and the combinatorial Proposition 4.19 below, which computes the series $Z^0_\Delta(q)$. $\square$

Finally, we investigate the higher rank equivariant case. There is a slight modification of the type A_n pattern where to each label, which are actually residue classes modulo $n+1$, a fixed modulo $n+1$ residue class a is added. This again induces a labelling on the Young diagrams which we will call this the *diagonal a-labelling*. More generally, we can consider l-tuples of such Young diagrams $\underline{\lambda}$. To any vector $\underline{a} \in (\mathbb{Z}/(n+1)\mathbb{Z})^l$ of length l consisting of residue classes modulo $n+1$ we associate the diagonal a_i-labelling on λ_i for each $1 \le i \le l$. The is called the diagonal $\underline{a}$-labelling (or, shortly, $\underline{a}$-labelling) of λ. As we will see immediately, the set of such tuples appears in the analysis of the higher rank moduli spaces $\mathcal{M}^r([\mathbb{C}^2/G])$.

EXAMPLE 4.7. For $n=2$ the diagonal $(2,1)$-coloring on the Young diagrams corresponding to the pair of partitions $((4,3,2),(2,1,1,1))$ is the following:

$$\left(\begin{array}{c} \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 0 & 1 & \\ \hline 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \hline \end{array} \quad , \quad \begin{array}{|c|c|c|c|} \hline 0 & & & \\ \hline 1 & 2 & 0 & 1 \\ \hline \end{array} \end{array} \right).$$

We will denote the set of Young diagrams with the a-labelling as $\mathcal{Z}_{\Delta(a)}$, and the set of l-tuples of Young diagrams with the $\underline{a}$-labelling as $\mathcal{Z}_{\Delta(\underline{a})}$ for any $\underline{a} \in (\mathbb{Z}/(n+1)\mathbb{Z})^l$. For any $0 \le i \le n$ and $\underline{\lambda} \in \mathcal{Z}_{\Delta(\underline{a})}$ we denote by $\mathrm{wt}_i(\underline{a}, \underline{\lambda})$, the *i-weight of $\underline{\lambda}$*, which is the number of boxes in $\underline{\lambda}$, whose label according to the $\underline{a}$-labelling is i. Clearly,

$\sum_{i=0}^{n} \mathrm{wt}_i(\underline{a}, \underline{\lambda}) = |\underline{\lambda}|$ for any $\underline{a} \in (\mathbb{Z}/(n+1)\mathbb{Z})^l$. We arrange the the i-weights into a vector $\underline{\mathrm{wt}}(\underline{a}, \underline{\lambda}) = (\mathrm{wt}_0(\underline{a}), \underline{\lambda}, \ldots, \mathrm{wt}_n(\underline{a}, \underline{\lambda})) \in (\mathbb{Z}_{\geq 0})^{n+1}$.

The colored (multivariable) generating series of a-labelled Young diagrams is defined as

$$Z_{\Delta(a)}(\underline{q}) = \sum_{\lambda \in \mathcal{Z}_\Delta(a)} q^{\underline{\mathrm{wt}}(a,Y)} \ .$$

Similarly, the colored (multivariable) generating series of l-tuples of colored Young diagrams is defined as

$$Z_{\Delta(\underline{a})}(\underline{q}) = \sum_{\underline{\lambda} \in \mathcal{Z}_\Delta(\underline{a})} q^{\underline{\mathrm{wt}}(\underline{a},\underline{\lambda})} \ .$$

Since tranforming the 0-labelling to the a-labelling corresponds to a change of variables $q_i \to q_{i+a}$, we obtain from Proposition 4.4

COROLLARY 4.8. *The generating series of a-labelled partitions has the following form:*

$$(4.4) \qquad Z_{\Delta(a)}(q_0, \ldots, q_n) = \frac{\sum_{\overline{m}=(m_1,\ldots,m_n)\in\mathbb{Z}^k}^{\infty} q_{1+a}^{m_1} \cdot \cdots \cdot q_{n+a}^{m_n} (q^{1/2})^{\overline{m}^\top \cdot C \cdot \overline{m}}}{\prod_{m=1}^{\infty} (1-q^m)^{n+1}},$$

where $q = q_0 \cdot \cdots \cdot q_n$ and C is the (finite) Cartan matrix of type A_n.

As a set $\mathcal{Z}_\Delta(\underline{a}) = \prod_{m=1}^{l} \mathcal{Z}_\Delta(a_m)$. Therefore, one immediately obtains

COROLLARY 4.9.

$$Z_{\Delta(\underline{a})}(\underline{q}) = \prod_{m=1}^{l} Z_{\Delta(a_m)}(\underline{q}) \ .$$

In particular, the calculation of $Z_{\Delta(\underline{a})}(\underline{q})$ is easily reduced to that of $Z_{\Delta(a)}(\underline{q})$.

Let us fix a dimension vector $\underline{w}$. It can be shown that there is a $T = (\mathbb{C}^*)^{|\underline{w}|+2}$-action on the associated quiver varieties $\mathcal{M}(\underline{v}, \underline{w})$ for all $\underline{v}$, whose fixed points are isolated. Let $\underline{a} = (0, \ldots, 0, \ldots, n-1, \ldots, n-1)$, where for each $c \in C$ the number of c's in $\underline{a}$ is w_c. Elements of $\underline{a}$ correspond in turn to basis vectors of W. The exact order of the entries is not important for us since a permutation of them corresponds to an automorphism of W on which, as mentioned above, the topology of $\mathcal{M}(\underline{v}, \underline{w})$ does not depend.

PROPOSITION 4.10. *[59, Proposition 5.7] The T-fixed points of $\mathcal{M}(\underline{v}, \underline{w})$ are indexed by $|\underline{w}|$-tuples of diagonally colored Young diagrams $\underline{\lambda}$ such that $|\underline{\lambda}| = |\underline{v}|$, the j-th diagram λ_j in $\underline{\lambda}$ is given the a_j-coloring, and $\underline{\mathrm{wt}}(\underline{a}, \underline{\lambda}) = \underline{v}$.*

COROLLARY 4.11.

$$Z_\Delta^W(\underline{q}) = \sum_{\underline{v}} \chi(\mathcal{M}(\underline{v}, \underline{w})) \underline{q}^{\underline{v}} = Z_{\Delta(\underline{a})}(\underline{q}) \ .$$

PROOF OF THEOREM 3.9. Putting together Corollary 4.8, Corollary 4.11 and Corollary 4.9 gives the result. $\square$

4.3. First approach: generalized Frobenius partitions

In this section we introduce a combinatorial method inspired by [12] generalizing the ideas of Andrews [2] and we give a proof of Proposition 4.4 as developed in [26].

DEFINITION 4.12. Two rows of nonnegative integers

$$
\begin{pmatrix}
f_1 & f_2 & \cdots & f_d \\
g_1 & g_2 & \cdots & g_d
\end{pmatrix}
$$

are called a *generalized Frobenius partition* or *F-partition* of k if

$$
k = d + \sum_{i=1}^{d}(f_i + g_i) \,.
$$

REMARK 4.13. A generalized Frobenius partition is a classical Frobenius partition if moreover $f_1 > f_2 > \cdots > f_d \geq 0$ and $g_1 > g_2 > \cdots > g_d \geq 0$. In this case, we can associate to the F-partition a Young diagram from which if we delete the d long diagonal then the lengths of the rows below it are f_1, f_2, etc. and the length of the columns above the diagonal are g_1, g_2, etc. This correspondence between Young diagrams and classical F-partitions is bijective.

Let H be an arbitrary set consisting of finite sequences of nonnegative integer. For arbitrary integers d and k let $P_H(k,d)$ denote the sequences in H of length d which sum up to k. For any pair of such sets H_1 and H_2, let moreover $P_{H_1,H_2}(k)$ be the number of generalized Frobenius partitions of k with elements in the first row $(f_1, \ldots, f_r)$ from H_1 and with elements in the second row $(g_1, \ldots, g_d)$ from H_2. Then the very useful result of Andrews says the following

THEOREM 4.14 ([2], Section 3).

$$
\sum_{k=0}^{\infty} P_{H_1,H_2}(k)q^k = [z^0] \sum_{k,m} P_{H_1}(k,d)q^k(zq)^d \sum_{k,d} P_{H_2}(k,d)q^k z^{-d} \,,
$$

where $[z^m]\sum A_k z^k = A_m$.

The term q^d in the first term of the right hand size corresponds to the contribution of the diagonals. To have a more symmetric formula we will slightly change the notions. Transform each generalized Frobenius partition

$$
\begin{pmatrix}
f_1 & f_2 & \cdots & f_d \\
g_1 & g_2 & \cdots & g_d
\end{pmatrix}
$$

into

$$
\begin{pmatrix}
f_1+1 & f_2+1 & \cdots & f_d+1 \\
g_1 & g_2 & \cdots & g_d
\end{pmatrix} \,.
$$

Then

$$
k = \sum_{i=1}^{d}((f_i+1) + g_i) \,.
$$

For H an arbitrary set of sequences as above, let $H' = \{(f_1 + 1, \ldots, f_d + 1) :$ $(f_1, \ldots, f_d) \in H\}$ be the *shift* of H by one upward. Then $P_{H'}(k, d) = P_H(k - d, d)$, and

$$(4.5) \qquad \sum_{k=0}^{\infty} P_{H_1', H_2}(k) q^k = [z^0] \sum_{k,d} P_{H_1'}(k, d) q^k z^d \sum_{k,d} P_{H_2}(k, d) q^k z^{-d} .$$

We aim for a multivariable generalization of Theorem 4.14. To be as general as possible we introduce temporarily an arbitrary labelling (or coloring) set C. We will specialize to the case of $C = \mathbb{Z}/(n + 1)\mathbb{Z}$ only later.

Let $\underline{k}$ be a vector of nonnegative integers indexed by the elements of C.

DEFINITION 4.15. Two series of vectors consisting of integers and arranged into two rows as

$$\begin{pmatrix} \underline{f}_1 & \underline{f}_2 & \cdots & \underline{f}_d \\ \underline{g}_1 & \underline{g}_2 & \cdots & \underline{g}_d \end{pmatrix}$$

are called a *colored generalized Frobenius partition* or *colored F-partition* of $\underline{k}$ if

(1) the elements in $\underline{f}_i$ and $\underline{g}_i$ are indexed by the elements $c \in C$ for each $1 \leq i \leq d$;

(2) $f_{i,c} \geq 0$ and $g_{i,c} \geq 0$ for every $1 \leq i \leq d$ and $c \in C$;

(3) $\sum_{i=1}^{d} (f_{i,c} + g_{i,c}) = k_c$ for each $c \in C$.

We will call $k = \sum_{c \in C} k_c = \sum_{c \in C} \sum_{i=1}^{d} (f_{i,c} + g_{i,c})$ the *total weight* of such a colored F-partition.

At the moment we do not require any further relations between the elements $f_{i,c}$ and $g_{i,c}$ but see end of this section and particularly Example 4.17 below, where we apply this general construction to the enumeration of diagonally colored Young diagrams.

Let H be an arbitrary set consisting of tuples of vectors, each of which is indexed by elements of C. For an arbitrary vector $\underline{k}$ indexed by the elements of C and consisting of nonnegative integers let $P_H(\underline{k}, d)$ be the number of d-tuples of vectors $(\underline{f}_1, \ldots, \underline{f}_d) \in H$ which satisfy conditions (1) and (2) such that $\sum_{i=1}^{d} f_{i,c} = k_c$. If both H_1 and H_2 are sets consisting of tuples of vectors, each element of which is indexed by elements of C, then let $P_{H_1,H_2}(\underline{k})$ be the number of colored F-partition of $\underline{k}$ in which the top row is in H_1 and the bottom row is in H_2.

Then the same ideas as that of Theorem 4.14 imply the following multivariable analogue of (4.5).

THEOREM 4.16.

$$\sum_{\underline{k}} P_{H_1,H_2}(\underline{k}) \underline{q}^{\underline{k}} = [z^0] \sum_{\underline{k},d} P_{H_1}(\underline{k}, d) z^d \underline{q}^{\underline{k}} \sum_{\underline{k},d} P_{H_2}(\underline{k}, d) z^{-d} \underline{q}^{\underline{k}} .$$

We return finally to the setting of the orbifold Hilbert scheme, whose T-fixed points are in bijection with diagonally labelled diagrams as explained in the beginning

of 4.2. We let $C = \mathbb{Z}/(n+1)\mathbb{Z}$ be the set of labels which can appear in the pattern of type A_n. To be able to apply Theorem 4.16 we first associate to each Young diagram $\lambda \in \mathcal{Z}_\Delta$ a colored F-partition of $\underline{k} = \{\mathrm{wt}_c(Y)\}_{c \in C}$ which uniquely describes the diagonal coloring on λ. Assume that the main diagonal of Y consists of d blocks. Then the associated colored F-partition is

$$
\begin{pmatrix} \underline{f}_1 & \underline{f}_2 & \cdots & \underline{f}_d \\ \underline{g}_1 & \underline{g}_2 & \cdots & \underline{g}_d \end{pmatrix} ,
$$

where $f_{i,c}$ is the number of blocks of color c in the i-th row below and including the main diagonal, and $g_{i,c}$ is the number of blocks of color c in the i-th column above the main diagonal for every $1 \le i \le d$.

EXAMPLE 4.17. The colored F-partition associated to the first diagram in Example 4.7 is

$$
\begin{pmatrix} (1,1,1) & (1,0,1) \\ (1,1,1) & (0,1,0) \end{pmatrix} ,
$$

where each $\underline{f}_i = (f_{i,0}, f_{i,1}, f_{i,2})$ and each $\underline{g}_i = (g_{i,0}, g_{i,1}, g_{i,2})$.

For $i = 1, 2$ let H_i be the set of tuples of vectors which can appear as the i-th row of a colored F-partition associated to a diagonally labelled Young diagram in the above construction. Then

$$
(4.6) \qquad Z_{\Delta,a}(\underline{q}) = \sum_{\underline{k}} P_{H_1,H_2}(\underline{k}) \underline{q}^{\underline{k}} .
$$

LEMMA 4.18.

(1)

$$
\sum_{\underline{k}} P_{H_1}(\underline{k}, d) z^d \underline{q}^{\underline{k}} = \prod_{k=0}^{\infty} \prod_{i=0}^{n} (1 + z q_0 \ldots q_i q^k) .
$$

(2)

$$
\sum_{\underline{k}} P_{H_2}(\underline{k}, d) z^d \underline{q}^{\underline{k}} = \prod_{k=0}^{\infty} \prod_{i=0}^{n} (1 + z^{-1} q_{i+1} \ldots q_n q^k) .
$$

PROOF. (1) It is clear that each term $z q_0 \ldots q_i q^k$ corresponds to a part of a column above and including the main diagonal which has length $(n+1)k + i$. Conversely, the decomposition of each nonnegative number as $(n+1)k + i$ is unique.

The proof of (2) is similar. □

The product of the two generating series in Lemma 4.18 is

$$\sum_{\underline{k}} P_{H_1}(\underline{k}, d) z^d \underline{q}^{\underline{k}} \cdot \sum_{\underline{k}} P_{H_2}(\underline{k}, d) z^d \underline{q}^{\underline{k}}$$

$$= \prod_{k=0}^{\infty} \prod_{i=0}^{n} (1 + z q_0 \ldots q_i q^k)(1 + z^{-1} q_{i+1} \ldots q_n q^k)$$

(4.7)

$$= \prod_{k=1}^{\infty} \prod_{i=0}^{n} (1 + z q_{i+1}^{-1} \ldots q_n^{-1} q^k)(1 + (z q_{i+1}^{-1} \ldots q_n^{-1})^{-1} q^{k-1})$$

$$= \left(\prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^n \prod_{i=0}^{n} \left(\sum_{j_i=-\infty}^{\infty} (z q_{i+1}^{-1} \ldots q_n^{-1})^{j_i} q^{\binom{j_i+1}{2}} \right) ,$$

where at the last equality we have used the following form of the Jacobi triple product formula:

$$\prod_{n=1}^{\infty} (1 + z q^n)(1 + z^{-1} q^{n-1}) = \left(\prod_{n=1}^{\infty} (1 - q^n)^{-1} \right) \sum_{j=-\infty}^{\infty} z^j q^{\binom{j+1}{2}} .$$

By (4.6) and Theorem 4.16 to obtain $Z_\Delta(\underline{q})$ we have to calculate the coefficient of z^0 in (4.7).

(4.8)

$$Z_\Delta(\underline{q}) = [z^0] \left(\prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{n+1} \prod_{i=0}^{n} \left(\sum_{j_i=-\infty}^{\infty} (z q_{i+1}^{-1} \ldots q_n^{-1})^{j_i} q^{\binom{j_i+1}{2}} \right)$$

$$= \left(\prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{n+1} \cdot \sum_{\substack{\underline{j}=(j_0,\ldots,j_n)\in\mathbb{Z}^{n+1} \\ \sum_i j_i = 0}} q_1^{-j_0} \cdot \ldots \cdot q_n^{-j_0-\cdots-j_{n-1}} q^{\sum_{i=0}^{n} \binom{j_i+1}{2}} .$$

Let us introduce the following series of integers:

$$\begin{aligned}
m_1 &= -j_0 , \\
m_2 &= -j_0 - j_1 , \\
&\vdots \\
m_n &= -j_0 - j_1 - \cdots - j_{n-1} .
\end{aligned}$$

It is obvious that the map

$$\begin{aligned}
\{(j_0, \ldots, j_n) \in \mathbb{Z}^{n+1} \ : \ \textstyle\sum_i j_i = 0\} &\to \mathbb{Z}^n \\
(j_0, \ldots, j_n) &\mapsto (m_1, \ldots, m_n)
\end{aligned}$$

is a bijection. The inverse of it is

$$
\begin{aligned}
j_0 &= -m_1 \, , \\
j_1 &= -m_2 + m_1 \, , \\
&\;\;\vdots \\
j_{n-1} &= -m_n + m_{n-1} \, , \\
j_n &= m_n \, .
\end{aligned}
$$

If $n = 1$, then

$$
(4.9) \qquad \sum_{i=0}^{1} \binom{j_i + 1}{2} = \binom{-m_1 + 1}{2} + \binom{m_1 + 1}{2} = m_1^2 = \frac{1}{2} \left(\underline{m}^\top \cdot C \cdot \underline{m} \right) \, ,
$$

where $C = (2)$ is the Cartan matrix of type A_1.

If $n > 1$, then

$$
\begin{aligned}
(4.10) \qquad \sum_{i=0}^{n} \binom{j_i + 1}{2} &= \binom{-m_1 + 1}{2} + \sum_{i=1}^{n-1} \binom{-m_{i+1} + m_i + 1}{2} + \binom{m_n + 1}{2} \\
&= m_1^2 + \cdots + m_n^2 - \sum_{i=1}^{n-1} m_i m_{i+1} \\
&= \frac{1}{2} \left(\underline{m}^\top \cdot C \cdot \underline{m} \right) \, ,
\end{aligned}
$$

where

$$
C = \begin{pmatrix}
2 & -1 & & & & \\
-1 & 2 & -1 & & & \\
& -1 & 2 & & & \\
& & & \ddots & & \\
& & & & & -1 \\
& & & & -1 & 2
\end{pmatrix}
$$

is the Cartan matrix of type A_n.

Equations (4.8), (4.9) and (4.10) together immediately imply Proposition 4.4 for all $n > 0$.

4.4. Second approach: abacus configurations

We now introduce a seemingly independent but also standard combinatorics related to the type A root system, which will allow us to relate the generating series Z_Δ of diagonally labelled partitions to the specialized series Z_Δ^0 of 0-generated partitions. We follow the notations of [44].

The *abacus of type A_n* is the arrangement of the set of integers in $(n+1)$ columns according to the following pattern.

$$
\begin{array}{cccccc}
\vdots & \vdots & & \vdots & \vdots \\
-2n-1 & -2n & \ldots & -n-2 & -n-1 \\
-n & -n+1 & \ldots & -1 & 0 \\
1 & 2 & \ldots & n & n+1 \\
n+2 & n+3 & \ldots & 2n+1 & 2n+2 \\
\vdots & \vdots & & \vdots & \vdots
\end{array}
$$

Each integer in this pattern is called a *position*. For any integer $1 \le k \le n+1$ the set of positions in the k-th column of the abacus is called the k-th *runner*. An *abacus configuration* is a set of *beads*, denoted by $\bigcirc$, placed on the positions, with each position occupied by at most one bead.

To an diagonally labelled partition $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{Z}_\Delta$ we associate its *abacus representation* (sometimes also called *Maya diagram*) as follows: place a bead in position $\lambda_i - i + 1$ for all i, interpreting λ_i as 0 for $i > k$. Alternatively, the abacus representation can be described by tracing the outer profile of the Young diagram of a partition: the occupied positions occur where the profile moves "down", whereas the empty positions are where the profile moves "right". In the abacus representation of a partition, the number of occupied positive positions is always equal to the number of absent nonpositive positions; we call such abacus configurations *balanced*. Conversely, it is easy to see that any balanced configuration represents a unique diagonally labelled partition, an element of $\mathcal{Z}_\Delta$.

For $n = 0$, we obtain a representation of partitions on a single runner; this is sometimes called the *Dirac sea* representation of partitions.

The $(n+1)$-*core* of a labelled partition $\lambda \in \mathcal{Z}_\Delta$ is the partition obtained from λ by successively removing border strips of length $n+1$, leaving a partition at each step, until this is no longer possible. Here a *border strip* is a skew Young diagram which does not contain 2×2 blocks and which contains exactly one j-labelled block for all labels j. The removal of a border strip corresponds in the abacus representation to shifting one of the beads up on its runner, if there is an empty space on the runner above it. In this way, the core of a partition corresponds to the bead configuration in which all the beads are shifted up as much as possible; this in particular shows that the $(n+1)$-core of a partition is well-defined. We denote by $\mathcal{C}_\Delta$ the set of $(n+1)$-core partitions, and

$$
c \colon \mathcal{Z}_\Delta \to \mathcal{C}_\Delta
$$

the map which takes an diagonally labelled partition to its $(n+1)$-core.

Given an $(n+1)$-core λ, we can read the $(n+1)$ runners of its abacus rrepresentation separately. These will not necessarily be balanced. The i-th one will be shifted from the balanced position by a certain integer number a_i steps, which is negative if the shift is toward the negative positions (upwards), and positive otherwise. These numbers satisfy $\sum_{i=0}^{n} a_i = 0$, since the original abacus configuration was balanced.

The set $\{a_1, \ldots, a_n\}$ completely determines the partition, so we get a bijection

$$(4.11) \qquad \mathcal{C}_\Delta \longleftrightarrow \left\{\sum_{i=0}^n a_i = 0\right\} \subset \mathbb{Z}^{n+1}.$$

We will represent an $(n+1)$-core partition by the corresponding $(n+1)$-tuple $\underline{a} = (a_0, \ldots, a_n)$.

On the other hand, for an arbitrary partition, on each runner we have a partition up to shift, so we get a bijection

$$\mathcal{Z}_\Delta \longleftrightarrow \mathcal{C}_\Delta \times \mathcal{P}^{n+1}.$$

This corresponds to the structure of formula (4.2) above; its denominator is the generating series of $(n+1)$-tuples of (unlabelled) partitions, whereas its numerator (after eliminating a variable) is exactly a sum over $\underline{a} \in \mathcal{C}_\Delta$. The multiweight of a core partition corresponding to an element $\underline{a}$ is given by the quadratic expression $Q(\underline{a})$ in the exponent of the numerator of (4.2). For more details, see Bijections 1-2 in [19, §2].

Our purpose in the remaining part of this section is to prove the following, completely combinatorial statement.

PROPOSITION 4.19. *Let Δ be of type A_n, and let ξ be a primitive $(n+2)$-nd root of unity. Then the generating series of 0-generated partitions can be computed from that of all diagonally labelled ones by the following substitution:*

$$Z_\Delta^0(q) = Z_\Delta(q_0, \ldots, q_n)\Big|_{q_0 = \xi^{-n}q, q_1 = \cdots = q_n = \xi}.$$

We start by combinatorially relating partitions to 0-generated partitions. $\mathcal{Z}_\Delta^0$ is clearly a subset of $\mathcal{Z}_\Delta$, but there is also a map

$$p \colon \mathcal{Z}_\Delta \to \mathcal{Z}_\Delta^0$$

defined as follows: for an arbitrary partition λ, let $p(\lambda)$ be the smallest 0-generated partition containing it. Since the set of 0-generated partitions is closed under intersection, $p(\lambda)$ is well-defined, and it can be constructed as follows: $p(\lambda)$ is the complement of the unions of the translates of $\mathbb{N} \times \mathbb{N}$ to 0-labelled blocks in the complement of λ. It is clear that $p(\lambda)$ can equivalently be obtained by adding all possible addable blocks to λ of labels different from 0.

REMARK 4.20. The map p can also be described in the language of ideals. If the monomial ideal $I \lhd \mathbb{C}[x, y]$ corresponds to the partition λ, then the monomial ideal $i^* p_* I = (I \cap \mathbb{C}[x, y]^{G_\Delta}).\mathbb{C}[x, y] \lhd \mathbb{C}[x, y]$ corresponds to the partition $p(\lambda)$.

LEMMA 4.21. *The bead configurations corresponding to 0-generated partitions are exactly those which have all beads right-justified on each row, with no empty position*

to the right of a filled position. The map $p\colon \mathcal{Z}_\Delta \to \mathcal{Z}_\Delta^0$ can be described in the abacus representation by the process of pushing all beads of an abacus configuration as far right as possible.

PROOF. This follows from the description of the map from a partition to its abacus representation using the profile of the partition. Indeed, a 0-generated partition has a profile which only turns from "down" to "right" at 0-labelled blocks. In other words, the only time when a string of filled positions can be followed by an empty position is when the last filled position is on the rightmost runner. In other words, there cannot be empty positions to the right of filled positions in a row. The proof of the second statement is similar. □

REMARK 4.22. As explained above, the maps $c\colon \mathcal{Z}_\Delta \to \mathcal{C}_\Delta$ and $p\colon \mathcal{Z}_\Delta \to \mathcal{Z}_\Delta^0$ have natural descriptions on abacus configurations: c corresponds to pushing beads all the way up within their column, whereas p corresponds to pushing beads all the way to the right within their row. It is then clear that there is also a third map $\mathcal{Z}_\Delta \to {}^0\mathcal{Z}_\Delta \subset \mathcal{Z}_\Delta$, dual to p, defined on the abacus by pushing beads all the way to the left. On labelled partitions this corresponds to the operation of removing all possible blocks with labels different from 0. This dual constuction occured in the literature earlier in [20].

PROOF OF PROPOSITION 4.19. We will prove the substitution formula on the fibres of the map $p\colon \mathcal{Z}_\Delta \to \mathcal{Z}_\Delta^0$. In other words, we need to show that for any given $\lambda_0 \in \mathcal{Z}_\Delta^0$, we have

$$(4.12) \qquad \sum_{\mu \in p^{-1}(\lambda_0)} \underline{q}^{\underline{\mathrm{wt}}(\mu)}\bigg|_{q_1=\cdots=q_n=\xi,\, q_0=\xi^{-n}q} = q^{\mathrm{wt}_0(\lambda_0)}.$$

As a first step, we reduce the computation to 0-generated cores. Given an arbitrary 0-generated partition λ, by the first part of Lemma 4.21 its core $\nu = c(\lambda)$ is also 0-generated, and the corresponding abacus configuration can be obtained by permuting the rows of the configuration of λ. Fix one such permutation σ of the rows. Then, using the second part of Lemma 4.21, we can use the row permutation σ to define a bijection

$$\tilde{\sigma}\colon p^{-1}(\lambda) \to p^{-1}(\nu)$$

between (abacus representations of) partitions in the fibres, mapping λ itself to ν.

The difference between the partitions λ and ν is a certain number of border strips, each removal represented by pushing up one bead on some runner by one step. Each border strip contains one block of each label, so the total number of times we need to push up a bead by one step on the different runners is $N = \mathrm{wt}_0(\lambda) - \mathrm{wt}_0(\nu)$. Thus, with $q = q_0 \cdot \ldots \cdot q_n$ as in the substitution above, we can write

$$\underline{q}^{\underline{\mathrm{wt}}(\lambda)} = q^{\mathrm{wt}_0(\lambda)-\mathrm{wt}_0(\nu)}\underline{q}^{\underline{\mathrm{wt}}(\nu)}.$$

On the other hand, it is easy to see that in fact for any $\mu \in p^{-1}(\lambda)$, the corresponding $\tilde{\sigma}(\mu)$ can also be obtained by pushing up beads exactly N times, one step at a time, the difference being just in the runners on which these shifts are performed. This means that each μ differs from $\tilde{\sigma}(\mu)$ by the same number $N = \mathrm{wt}(\lambda) - \mathrm{wt}(\nu)$ of border strips. Therefore, we have

$$\sum_{\mu \in p^{-1}(\lambda)} q^{\underline{\mathrm{wt}}(\mu)} = q^{\mathrm{wt}_0(\lambda) - \mathrm{wt}_0(\nu)} \sum_{\mu \in p^{-1}(\nu)} q^{\underline{\mathrm{wt}}(\mu)}.$$

This is clearly compatible with (4.12) and reduces the argument to 0-generated core partitions.

Fix a 0-generated core $\lambda \in \mathcal{Z}_\Delta^0 \cap \mathcal{C}_\Delta$; using Lemma 4.21 again, the corresponding $(n+1)$-tuple is a set of *nondecreasing* integers $\underline{a} = (a_0, \ldots, a_n)$ summing to 0. The fibre $p^{-1}(\lambda)$ consists of partitions whose abacus representation contains the same number of beads in each row as λ. The shift of one bead to the left results in the removal in the partition of a block labelled i, with $1 \le i \le n$. After substitution, this multiplies the contribution of the diagram on the right hand side of (4.12) by ξ^{-1}. If we fix all but one row, which contains k beads, then these contributions add up to

$$\sum_{n_1=0}^{n-k+1} \sum_{n_2=0}^{n_1} \cdots \sum_{n_k=0}^{n_{k-1}} (\xi^{-1})^{n_1 + \cdots + n_k} = \binom{n+1}{k}_{\xi^{-1}},$$

where $\binom{m}{r}_z = \frac{[m]_z!}{[r]_z![m-r]_z!}$ is the Gaussian binomial coefficient, with $[m]_z = \frac{1-z^m}{1-z}$.

The number of rows containing exactly k beads in the configuration corresponding to λ is $a_{n+1-k} - a_{n-k}$. Therefore, the total contribution of the preimages, the left hand side of (4.12), is

$$\sum_{\mu \in p^{-1}(\lambda)} q^{\underline{\mathrm{wt}}(\mu)} \Bigg|_{q_1 = \cdots = q_n = \xi, q_0 = \xi^{-n} q}$$

$$= \prod_{k=1}^{n} \binom{n+1}{k}_{\xi^{-1}}^{a_{n+1-k} - a_{n-k}} q^{\underline{\mathrm{wt}}(\lambda)} \Bigg|_{q_1 = \cdots = q_n = \xi, q_0 = \xi^{-n} q}$$

$$= \prod_{l=0}^{n} \left(\frac{\binom{n+1}{n+1-l}_{\xi^{-1}}}{\binom{n+1}{n-l}_{\xi^{-1}}} \right)^{a_l} q^{\underline{\mathrm{wt}}(\lambda)} \Bigg|_{q_1 = \cdots = q_n = \xi, q_0 = \xi^{-n} q}$$

$$= \prod_{l=0}^{n} \left(\frac{1 - \xi^{-l-1}}{1 - \xi^{l-n-1}} \right)^{a_l} q^{\underline{\mathrm{wt}}(\lambda)} \Bigg|_{q_1 = \cdots = q_n = \xi, q_0 = \xi^{-n} q}$$

$$= \prod_{l=1}^{n} \left(\frac{1 - \xi^{-n-1}}{1 - \xi^{-1}} \frac{1 - \xi^{-l-1}}{1 - \xi^{l-n-1}} \right)^{a_l} q^{\underline{\mathrm{wt}}(\lambda)} \Bigg|_{q_1 = \cdots = q_n = \xi, q_0 = \xi^{-n} q}$$

$$= \xi^{-\sum_{l=1}^{n} l a_l} q^{\underline{\mathrm{wt}}(\lambda)} \Bigg|_{q_1 = \cdots = q_n = \xi, q_0 = \xi^{-n} q},$$

where in the second equality we used $\binom{n+1}{0}_z = \binom{n+1}{n+1}_z = 1$, in the penultimate equality we used $a_0 = -a_1 - \cdots - a_n$, and in the last equality we used

$$\frac{1 - \xi^{-n-1}}{1 - \xi^{-1}} \frac{1 - \xi^{-l-1}}{1 - \xi^{l-n-1}} = \xi^{-l},$$

which can be checked to hold for ξ a primitive $(n + 2)$-nd root of unity. Incidentally, as the multiplicative order of ξ is exactly $n + 2$, all the denominators appearing above are non-vanishing. Finally, according to [19, §2], we have

$$\underline{q}^{\underline{\mathrm{wt}}(\lambda)} = q^{\frac{Q(\underline{a})}{2}} q_1^{a_1 + \cdots + a_n} \cdot \ldots \cdot q_n^{a_n},$$

where again $q = q_0 \cdot \ldots \cdot q_n$ and $Q : \mathbb{Z}^n \to \mathbb{Z}$ is the quadratic form associated to C_Δ. Since q_0 appears only in q on the right hand side, it is clear that $\frac{Q(\underline{a})}{2} = \mathrm{wt}_0(\lambda)$. Hence,

$$q^{\frac{Q(\underline{a})}{2}} q_1^{a_1 + \cdots + a_n} \cdots q_n^{a_n} \Big|_{q_1 = \cdots = q_n = \xi} = q^{\mathrm{wt}_0(\lambda)} \xi^{\sum_{l=1}^{n} l a_l}.$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

4.5. Outlook: cyclic quotient singularities of type $(p, 1)$

This section is based on [27]. Recall the cyclic quotient singularities of type $(p, 1)$ from the end of 2.2. Let us fix the integer p once and for all. The surface singularity $X(p, 1)$ is again toric, i.e. it carries a $(\mathbb{C}^*)^2$-action with an isolated fixed point. On the level of regular functions the fixed points are the monomials in

$$A := H^0(\mathcal{O}_{X(p,1)}) = \mathbb{C}[x, y]^{\mathbb{Z}_p} \cong \mathbb{C}[x^p, x^{p-1}y, \ldots, xy^{p-1}, y^p].$$

The $(\mathbb{C}^*)^2$-action lifts to $\mathrm{Hilb}^n(X(p, 1))$ for each n. The action of $(\mathbb{C}^*)^2$ on $\mathrm{Hilb}^n(X(p, 1))$ again has only isolated fixed points which are given by the finite colength monomial ideals in A.

As mentioned before, finite colength monomial ideals inside $\mathbb{C}[x, y]$ are in one-to-one correspondence with partitions and with Young diagrams. The generators of a monomial ideal are the functions corresponding to those blocks in the complement of the diagram which are at the corners. Since $A \subset \mathbb{C}[x, y]$, each fixed point of the $(\mathbb{C}^*)^2$-action on $\mathrm{Hilb}^n(X(p, 1))$ corresponds to again a Young diagram.

A block at position (i, j) is called a 0-*block* (for the singularity $X_{(p,1)}$) if $i + j \equiv 0 \pmod{n}$. We will call a Young diagram 0-*generated* (for the singularity $X_{(p,1)}$) if its generator blocks are 0-blocks. The 0-weight of a (not necessarily 0-generated) Young diagram λ is the number of 0-blocks inside the Young diagram. It is denoted as $\mathrm{wt}_0(\lambda)$. Let $\mathcal{P}_0$ be the set of 0-generated Young diagrams. It decomposes as

$$\mathcal{P}_0 = \bigsqcup_{n \geq 0} \mathcal{P}_0(n),$$

where $\mathcal{P}_0(n)$ is the set of 0-generated Young diagrams which have 0-weight n.

EXAMPLE 4.23. If $p = 3$, then the following is a 0-generated Young diagram (we also indicated the generating 0-blocks):
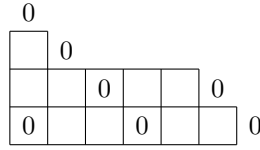
The 0-weight of this Young diagram is 3.

The generators of any ideal of A, when considered as functions in $\mathbb{C}[x, y]$, have to be invariant under the $\mathbb{Z}_p$-action. This implies the following statement.

LEMMA 4.24. *The monomial ideals inside A of colength n are in one-to-one correspondence with the 0-generated Young diagrams in $\mathcal{P}_0(n)$.*

COROLLARY 4.25.

$$Z_{X(p,1)}(q) = \sum_{\lambda \in \mathcal{P}_0} q^{\mathrm{wt}_0(\lambda)} = \sum_{n \geq 0} |\mathcal{P}_0(n)| q^n,$$

where $|S|$ denotes the number of elements in the set S.

For any 0-generated Young diagram, we can consider the area which is between the diagram and the line $x + y = c$. Here c is an integer congruent to 0 modulo p and it is as small as possible such that the line $x + y = c$ does not intersect the diagram. In other words, this line is just the antidiagonal closest to the diagram. The line cuts out the smallest isosceles right-angled triangle which contains the whole diagram. In the case of Example 4.23, this looks as follows:
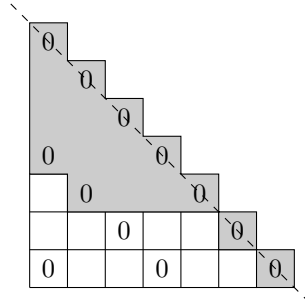
If, for example, $p = 1$ then the area between the diagram, the x and y coordinate axes, and the above mentioned antidiagonal, when rotated 45 degrees counterclockwise and flipped, is a special type of a *fountain of coins* as introduced in [58]. An (n, k) *fountain (of coins)* is an arrangement of n coins in rows such that there are exactly k consecutive coins in the bottom row, and such that each coin in a higher row touches exactly two coins in the next lower row. In our case, the 0-blocks in the area under consideration (which is colored grey in the diagram above) are replaced by coins or circles or zero symbols.

We generalize this notion to arbitrary p. An (n, k) *p-fountain* is an arrangement of n coins in rows such that

- there are exactly k consecutive coins in the bottom row,
- immediately below each coin there are exactly $p+1$ "descendant" coins in the next lower row,
- and p coins among the descendants of two neighboring coins coincide.

In other words, we rotate and flip the area between the axes and a specific antidiagonal, and the coins can be placed exactly on the 0-blocks, such that if there is a coin somewhere, then there has to be coins on all the 0-blocks in the area which have higher x or y coordinates in the original orientation of the plane. The empty diagram is considered as a $(0, 0)$ p-fountain.

Continuing the case of Example 4.23 further, the associated $(9, 7)$ 3-fountain is

$$\overset{\displaystyle 0 \quad 0}{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0}$$
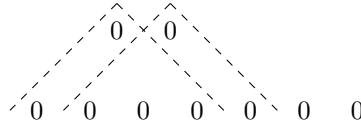
Here we also indicated the descendants of the coins in the upper row.

An (n, k) p-fountain is called primitive if its next-to-bottom row contains no empty positions, i.e. contains $k - p$ coins. In particular, the fountain with p coins in the bottom row but with no coin in the higher rows is primitive, but the ones with less than p coins in the bottom row are not primitive. The fountains that appear between our 0-generated Young diagrams and the diagonals are special because in each case there is at least one empty position in the next-to-bottom row, so they correspond exactly to the non-primitive p-fountains.

Let $f(n, k)$ (reps., $g(n, k)$) be the number of arbitrary (resp., primitive)(n, k) p-fountains. Let $F(q, z) = \sum_{n, k \geq 0} f(n, k) q^n z^k$ (resp., $G(q, z) = \sum_{n, k \geq 0} g(n, k) q^n z^k$) be the two variable generating function of the sequence $f(n, k)$ (resp., $g(n, k)$). We will calculate $F(q, z)$ and $G(q, z)$ by extending the ideas of [58].

By removing the bottom row of a primitive (n, k) p-fountain one obtains a $(n - k, k - p)$ p-fountain. Therefore,

$$g(n, k) = f(n - k, k - p) \qquad (n \geq k, k \geq p),$$

and

$$(4.13) \qquad\qquad G(q, z) = (qz)^p F(q, qz).$$

We prescribe that

$$(4.14) \quad f(0, 0) = \cdots = f(p - 1, p - 1) = 1 \text{ and } g(0, 0) = \cdots = g(p - 1, p - 1) = 0.$$

Let us consider an arbitrary (n, k) p-fountain $\mathcal{F}$, and assume that the first empty position in the next-to-bottom row is the r-th $(1 \leq r \leq k - p + 1)$. Then we can split $\mathcal{F}$ into a primitive $(m + p - 1, r + p - 1)$ p-fountain and a not necessarily primitive $(n - m, k - r)$ p-fountain after the first and before the last descendant coin of the

above mentioned missing r-th position. The descendant coins at the second to the penultimate positions will be doubled.

Sticking to our favorite Example 4.23, the splitting looks as follows:

$$
\begin{array}{ccccccccc}
0 & 0 & & & & & & 0 & 0 \\
0 & 0 & 0 & (\ 0 & 0 \ ) \ 0 & 0 & & & \\
\end{array}
\Rightarrow
\begin{array}{ccccc}
0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
+
\begin{array}{cccc}
0 & 0 & & \\
0 & 0 & 0 & 0 \\
\end{array}
$$

The dashed line on the left picture indicates the missing coin from the second row and its descendants. The primitive part is to the left of the right parenthesis, while the remaining part is to the right of the left parenthesis.

This factorization is unique.

$$(4.15) \quad f(n, k) = \sum_{\substack{0 \le m \le n-p+1 \\ 0 \le r \le k-p+1}} g(m + p - 1, r + p - 1) f(n - m, k - r) \qquad (n, k \ge p).$$

The conditions $m \le n - p + 1$ and $r \le k - p + 1$ are equivalent to $p - 1 \le n - m$ and $p - 1 \le k - r$ respectively. That is, the other remaining fountain has to have a first row of length at least $p - 1$. From (4.14) we see that $F(q, z)$ has the form

$$F(q, z) = 1 + qz + \cdots + (qz)^{p-1} + \ldots.$$

Then $F(q, z) - 1 - qz - \cdots - (qz)^{p-2}$ is exactly the generating function of fountains, which have a first row of length at least $p - 1$, that is, which can appear as the second factor on the right hand side of (4.15). Therefore,

$$(qz)^{-p+1} G(q, z)(F(q, z) - 1 - qz - \cdots - (qz)^{p-2})$$

enumerates all p-fountains for which $n, k \ge p$. The generating function of all p-fountains then satisfies

(4.16)
$$F(q, z) = 1 + qz + \cdots + (qz)^{p-1} + (qz)^{-p+1} G(q, z)(F(q, z) - 1 - qz - \cdots - (qz)^{p-2}).$$

Using (4.16),

$$(F(q, z) - 1 - qz - \cdots - (qz)^{p-2})(1 - (qz)^{-p+1} G(q, z)) = (qz)^{p-1}.$$

Then, by (4.13), the generating function of p-fountains is

$$
\begin{aligned}
(4.17) \qquad F(q, z) &= \frac{(qz)^{p-1}}{1 - (qz)^{-p+1} G(q, z)} + 1 + qz + \cdots + (qz)^{p-2} \\
&= \frac{(qz)^{p-1}}{1 - qz F(q, qz)} + \frac{1 - (qz)^{p-1}}{1 - qz} \\
&= \frac{(qz)^{p-1}}{1 - qz \left(\frac{(q^2 z)^{p-1}}{1 - q^2 z F(q, q^2 z)} + \frac{1 - (q^2 z)^{p-1}}{1 - q^2 z} \right)} + \frac{1 - (qz)^{p-1}}{1 - qz} = \cdots \\
&= \frac{(qz)^{p-1}}{1 - qz \left(\frac{(q^2 z)^{p-1}}{1 - q^2 z \left(\frac{(q^3 z)^{p-1}}{1 - \cdots} + \frac{1 - (q^3 z)^{p-1}}{1 - q^3 z} \right)} + \frac{1 - (q^2 z)^{p-1}}{1 - q^2 z} \right)} + \frac{1 - (qz)^{p-1}}{1 - qz}.
\end{aligned}
$$

Consequently, the generating function of primitive p-fountains is

$$(4.18) \quad G(q, z) = \frac{(q^2 z)^{p-1}}{1 - q^2 z \left(\frac{(q^3 z)^{p-1}}{1 - q^3 z \left(\frac{(q^4 z)^{p-1}}{1 - \ldots} + \frac{1 - (q^4 z)^{p-1}}{1 - q^4 z} \right)} + \frac{1 - (q^3 z)^{p-1}}{1 - q^3 z} \right)} + \frac{1 - (q^2 z)^{p-1}}{1 - q^2 z}.$$

REMARK 4.26. For $p = 1$, Ramanujan [3, p. 104] obtained the beautiful formula

$$F(q, z) = \frac{1}{1 - \frac{qz}{1 - \frac{q^2 z}{1 - \ldots}}} = \frac{\sum_{n \geq 0} (-qz)^n \frac{q^{n^2}}{(1-q)(1-q^2)\ldots(1-q^n)}}{\sum_{n \geq 0} (-z)^n \frac{q^{n^2}}{(1-q)(1-q^2)\ldots(1-q^n)}}.$$

The number of non-primitive (n, k) p-fountains is obviously

$$h(n, k) = f(n, k) - g(n, k),$$

which gives

$$(4.19) \qquad H(q, z) = F(q, z) - G(q, z) = F(q, z) - (qz)^p F(q, qz)$$

for the generating series $H(q, z)$ of the numbers $h(n, k)$.

PROOF OF THEOREM 3.10. As mentioned above, we augment each 0-generated Young diagram to the smallest isosceles right-angled triangle. The area between the diagram and the triangle will be a p-fountain. For a fixed p, the hypotenuse of the possible isosceles right-angled triangles contains $lp + 1$ blocks, where l is a non-negative integer. The number of 0-blocks in the triangle with $lp + 1$ blocks on the hypotenuse is $\sum_{0 \leq i \leq l} ip + 1 = p\frac{l(l+1)}{2} + l + 1$. Therefore, the two variable generating series of these triangles is

$$\sum_{l \geq 0} q^{p\frac{l(l+1)}{2} + l + 1} z^{lp + 1}.$$

We will see immediately that adding the terms with negative l will not affect the final result. Hence, we define

$$
\begin{aligned}
T(q, z) &= \sum_{l = -\infty}^{\infty} q^{p\frac{l(l+1)}{2} + l + 1} z^{lp + 1} = (qz) \sum_{l = -\infty}^{\infty} (q^p)^{\binom{l+1}{2}} (qz^p)^l \\
&= (qz) \prod_{n=1}^{\infty} (1 + z^p q^{np+1})(1 + z^{-p} q^{(n-1)p-1})(1 - q^{np}),
\end{aligned}
$$
$$(4.20)$$

where at the last equality we have used the following form of the Jacobi triple product identity:

$$\prod_{n=1}^{\infty} (1 + zq^n)(1 + z^{-1} q^{n-1})(1 - q^n) = \sum_{j = -\infty}^{\infty} z^j q^{\binom{j+1}{2}}.$$

The generating series of 0-generated Young diagrams is then

$$(4.21) \qquad \sum_{\lambda \in \mathcal{P}_0} q^{\mathrm{wt}_0(\lambda)} = [z^0] T(q, z) H(q^{-1}, z^{-1}).$$

Taking the coefficient of z^0 ensures that the hypotenuse of the triangle and the bottom row of the p-fountain match together, and also that the terms of $T(q, z)$ with negative powers of z do not contribute into the result. Putting together Corollary 4.25, (4.19)and (4.21) concludes the proof. $\qquad\square$

CHAPTER 5

Type D_n: ideals and Young walls

This chapter starts to develop the tools for proofs in the type D case. Young walls, which are the analogs of diagonally labelled Young diagrams of the type A case, are introduced. Their building blocks are shown to be in bijection with cells of the equivariant Grassmannian. Finally, with each invariant ideal a Young wall is associated.

5.1. The binary dihedral group

Fix an integer $n \geq 4$, and let Δ be the root system of type D_n. For ε a fixed primitive $(2n - 4)$-th root of unity, the corresponding subgroup G_Δ of $SL(2, \mathbb{C})$ can be generated by the following two elements σ and τ:

$$\sigma = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \qquad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The group G_Δ has order $4n - 8$, and is often called the binary dihedral group. We label its irreducible representations as shown in Table 1. There is a distinguished 2-dimensional representation, the defining representation $\rho_{\mathrm{nat}} = \rho_2$. See [35, 10] for more detailed information.

We will often meet the involution on the set of representations of G_Δ which is given by tensor product with the sign representation ρ_1: on the set of indices $\{0, \ldots, n\}$, this is the involution $j \mapsto \kappa(j)$ which swaps 0 and 1 and $n - 1$ and n, fixing other values $\{2, \ldots, n - 2\}$. Given $j \in \{0, \ldots, n\}$, we denote $\kappa(j, k) = \kappa^{kn}(j)$; this is an involution which is nontrivial when k and n are odd, and trivial otherwise. The special case $k = 1$ will also be denoted as $\underline{j} = \kappa^n(j)$.

The following identities will be useful:

(5.1) $\quad \rho_{n-1}^{\otimes 2} \cong \rho_n^{\otimes 2} \cong \rho_{\underline{0}}, \quad \rho_{n-1} \otimes \rho_n \cong \rho_{\underline{1}}, \quad \rho_1 \otimes \rho_{n-1} \cong \rho_n, \quad \rho_1 \otimes \rho_n \cong \rho_{n-1}, \quad \rho_1^{\otimes 2} \cong \rho_0.$

5.2. Young wall pattern and Young walls

We describe here the type D analogue of the set of labelled partitions used in type A, following [39, 41]. In this section, we only describe the combinatorics; see 2.4 for the representation-theoretic significance of this set.

ρ	$\mathrm{Tr}(1)$	$\mathrm{Tr}(\sigma)$	$\mathrm{Tr}(\tau)$
ρ_0	1	1	1
ρ_1	1	1	-1
ρ_2	2	$\varepsilon + \varepsilon^{-1}$	0
$\vdots$		$\vdots$	
ρ_{n-2}	2	$\varepsilon^{n-3} + \varepsilon^{-(n-3)}$	0
ρ_{n-1}	1	-1	$-i^n$
ρ_n	1	-1	i^n

TABLE 1. Labelling the representations of the group G_Δ

First we define the *Young wall pattern of type*[1] D_n, the analogue of the $(n+1)$-labelled positive quadrant lattice of type A_n used above. This is the following infinite pattern, consisting of two types of blocks: half-blocks carrying possible labels $j \in \{0, 1, n-1, n\}$, and full blocks carrying possible labels $1 < j < n-1$:
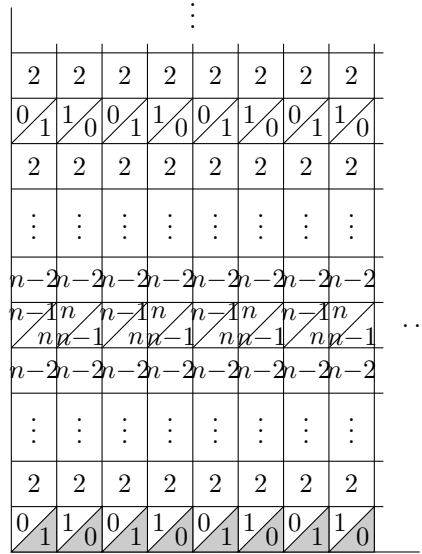
Next, we define the set of *Young walls*[2] *of type* D_n. A Young wall of type D_n is a subset Y of the infinite Young wall of type D_n, satisfying the following rules.

(YW1) Y contains all grey half-blocks, and a finite number of the white blocks and half-blocks.

(YW2) Y consists of continuous columns of blocks, with no block placed on top of a missing block or half-block.

---

[1] The combinatorics of this section should really be called type $\tilde{D}_n^{(1)}$, but we do not wish to overburden the notation. Also we have reflected the pattern in a vertical axis compared to the pictures of [39, 41].

[2] In [39, 41], these arrangements are called *proper Young walls*. Since we will not meet any other Young wall, we will drop the adjective *proper* for brevity.

(YW3) Except for the leftmost column, there are no free positions to the left of any block or half-block. Here the rows of half-blocks are thought of as two parallel rows; only half-blocks of the same orientation have to be present.

(YW4) A full column is a column with a full block or both half-blocks present at its top; then no two full columns have the same height[3].

Let $\mathcal{Z}_\Delta$ denote the set of all Young walls of type D_n. For any $Y \in \mathcal{Z}_\Delta$ and label $j \in \{0, \ldots, n\}$ let $wt_j(Y)$ be the number of white half-blocks, respectively blocks, of label j. These are collected into the multi-weight vector $\underline{wt}(Y) = (wt_0(Y), \ldots, wt_n(Y))$. The total weight of Y is the sum

$$|Y| = \sum_{j=0}^{n} wt_j(Y),$$

and for the formal variables $q_0, \ldots, q_n$,

$$\underline{q}^{\underline{wt}(Y)} = \prod_{j=0}^{n} q_j^{wt_j(Y)}.$$

5.3. Decomposition of $\mathbb{C}[x,y]$ and the transformed Young wall pattern

The group G_Δ acts on the affine plane $\mathbb{C}^2$ via the defining representation $\rho_{\text{nat}} = \rho_2$. Let $S = \mathbb{C}[x,y]$ be the coordinate ring of the plane, then $S = \oplus_{m \geq 0} S_m$ where S_m is the mth symmetric power of ρ_{nat}, the space of homogeneous polynomials of degree m of the coordinates x, y.

We further decompose

$$S_m = \bigoplus_{j=0}^{n} S_m[\rho_j]$$

into subrepresentations indexed by irreducible representations. We will also use this notation for linear subspaces: for $U \subset S_m$ a linear subspace, $U[\rho_j] = U \cap S_m[\rho_j]$. We will call an element $f \in S$ *degree homogeneous*, if $f \in S_m$ for some m; we call it *degree and weight homogeneous*, if $f \in S_m[\rho_j]$ for some m, j.

The decomposition of S into G_Δ-summands can be read off very conveniently from the *transformed Young wall pattern*. The transformation is an affine one, involving a shear: reflect the original Young wall pattern in the line $x = y$ in the plane, translate the nth row by n to the right, and remove the grey triangles of the original pattern. In this way, we get the following picture:

---

[3]This is the properness condition of [39].

As it can be checked readily, this is a representation of S and its decomposition into G_Δ-representations. The homogeneous components S_m are along the antidiagonals. For $1 < i < n-1$, a full block labelled j below the diagonal, together with its mirror image, correspond to a 2-dimensional representation ρ_j. For $j \in \{0, 1, n-1, n\}$, a full block labelled j on the diagonal, as well as a half-block labelled j below the diagonal with its mirror image, corresponds to a one-dimensional representation. The dimension of $S_m[\rho_j]$ is the same as the total number of full blocks labelled j on the mth diagonal in the transformed Young wall pattern, counting mirror images also.

It is easy to translate the conditions (YW1)-(YW4) into the combinatorics of the transformed pattern; see Proposition 5.7 and Remark 5.8 below. Pictures of some small Young walls in the transformed pattern can be found in Examples 6.5-6.9 below.

5.4. Subspaces and operators

For each non-negative integer m and irreducible representation ρ_j, consider the space $P_{m,j}$ of nontrivial G_Δ-invariant subspaces of minimal dimension in $S_m[\rho_j]$. Specifically, if ρ_j is one-dimensional, then these will be lines, and $P_{m,j}$ is simply the projectivization $\mathbb{P}S_m[\rho_j]$. If ρ_j is two-dimensional, then $P_{m,j}$ is a closed subvariety of $\mathrm{Gr}(2, S_m[\rho_j])$. It is easy to see that in this case also, $P_{m,j}$ is isomorphic to a projective space.

More generally, let $G^r_{m,j}$ be the space of $(r-1)$-dimensional projective subspaces of $P_{m,j}$. If ρ_j is one-dimensional, then this is the Grassmannian $\mathrm{Gr}(r, S_m[\rho_j])$. When ρ_j is two-dimensional, then $G^r_{m,j}$ is a closed subvariety of $\mathrm{Gr}(2r, S_m[\rho_j])$ isomorphic to a Grassmannian of rank r. Clearly $G^1_{m,j} = P_{m,j}$.

For $0 \le j \le n$, we introduce operators $L_j : \mathrm{Gr}(S) \to \mathrm{Gr}(S)$ on the Grassmannian $\mathrm{Gr}(S)$ of all linear subspaces of the vector space S as follows: for $v \in \mathrm{Gr}(S)$, we set

(1) $L_0 v = v$;
(2) $L_1 v = xy \cdot v$;
(3) for $1 < j < n-1$, $L_j v = \langle x^{j-1} \cdot v, y^{j-1} \cdot v \rangle$;
(4) $L_{n-1} v = (x^{n-2} - i^n y^{n-2}) \cdot v$;
(5) $L_n v = (x^{n-2} + i^n y^{n-2}) \cdot v$.

Sometimes we will use the notation $L_2 = L_{2,x} + L_{2,y}$ for the x- and y-component of the operator L_2, i.e. multiplication with x, respectively y. The operators above restrict to operators $L_0 \colon \mathrm{Gr}(S_m) \to \mathrm{Gr}(S_m)$, $L_1 \colon \mathrm{Gr}(S_m) \to \mathrm{Gr}(S_{m+2})$, $L_j \colon \mathrm{Gr}(S_m) \to \mathrm{Gr}(S_{m+j-1})$ for $1 < j < n-1$, and $L_{n-1}, L_n \colon \mathrm{Gr}(S_m) \to \mathrm{Gr}(S_{m+n-2})$ on the Grassmannians of the graded pieces S_m. To simplify notation, if we do not write the space to which these operators are applied, then application to $\langle 1 \rangle$ is meant. So, for example, the symbol L_1^2 standing alone denotes the vector subspace $\langle x^2 y^2 \rangle$ of S_4, while L_2 alone denotes the two-dimensional vector subspace $\langle x, y \rangle$ of S_1. For a linear subspace v of S, the sum $\sum_{j \in I} L_j v$ denotes the subspace of S generated by the images $L_j v$. We use the operator notation also for a set of subspaces; the meaning should be clear from the context.

5.5. Cell decompositions of equivariant Grassmannians

We start this section by defining decompositions of the Grassmannians $P_{m,j}$ of nontrivial G_Δ-invariant subspaces of minimal dimension in $S_m[\rho_j]$. Given (m, j), let $B_{m,j}$ denote the set of pairs of non-negative integers (k, l) such that $k + l = m$, $l \geq k$, and the block position (k, l) on the m-th antidiagonal on or below the main diagonal contains a block or half-block of color j. Here k is the row index, l is the column index, and both of them in a nonnegative integer. It clearly follows from this setup that

$$\dim P_{m,j} = |B_{m,j}| - 1.$$

PROPOSITION 5.1. *Given (m, j), there exists a locally closed stratification*

$$P_{m,j} = \bigsqcup_{(k,l) \in B_{m,j}} V_{k,l,j},$$

which is a standard stratification of the projective space $P_{m,j}$ into affine spaces $V_{k,l,j}$ of decreasing dimension.

We will call $V_{k,l,j}$ the *cells* of $P_{m,j}$. The decomposition will be defined inductively, based on the following Lemma. Recall that $j \mapsto \kappa(j)$ denotes the involution on $\{0, \ldots, n\}$ which swaps 0 and 1 and $n-1$ and n.

LEMMA 5.2. *For any $l \geq 0$ and any $j \in [0, n]$, we have an injection*

$$L_1 \colon P_{l-2,j} \to P_{l,\kappa(j)}.$$

This map is an isomorphism except in the case when the block or half-block in the bottom row of the transformed Young wall pattern on the l-th antidiagonal has label j, in which case the image has codimension one.

PROOF. It is clear that multiplication by L_1 induces an injection, so we simply need to check the dimensions. The statement then clearly follows by looking at the transformed Young wall pattern: multiplication by L_1 corresponds to shifting the

$(l-2)$-nd diagonal up by one diagonal step to the l-th diagonal; the number of blocks
or half-blocks labelled j is identical, unless the new (half-)block has label j, and then
the codimension is exactly one. $\square$

REMARK 5.3. The half-block in the bottom row of the transformed Young
wall pattern in the l-th antidiagonal has label $j = 0, 1$ for $l \equiv 0 \mod (2n-4)$
except at $(0, 0)$ where only 0 occurs. Half-blocks labelled $j = (n-1), n$ occur for
$l \equiv n-2 \mod (2n-4)$. For $j \in [2, n-2]$, there are full blocks labelled j in the
bottom row on antidiagonals for $l \equiv j-1$ or $2n-3-j \mod (2n-4)$.

PROOF OF PROPOSITION 5.1. Nontrivial cells $V_{0,l,j}$ need to be defined exactly
when the block or half-block in the bottom row of the transformed Young wall
pattern in the l-th antidiagonal has label j. In these cases, we set the cells along the
bottom row to be

$$V_{0,l,j} = P_{l,j} \setminus L_1 P_{l-2,\kappa(j)}.$$

Once the cells $V_{0,k,j}$ along the bottom row are defined, we define the general cells for
all $0 \leq j \leq n$, all l and k by

$$V_{k,k+l,j} = L_1^k V_{0,l,\kappa^k(j)}.$$

What this says is that the cells are shifted up diagonally by L_1, taking into account
that L_1 multiplies by the sign representation, so shifts the indices by the appropriate
power of the involution κ. By induction, we obtain a decomposition of $P_{m,j}$ with the
stated properties. $\square$

As it is well known, a decomposition of a projectivization of a vector space into
affine cells is equivalent to giving a flag in the space itself. This induces a natural
decomposition of all higher rank Grassmannians into *Schubert cells*, which are known
to be affine. Thus our cell decomposition of $P_{m,j}$ induces cell decompositions of all
$G_{m,j}^r$. Since the cells in the first decomposition are indexed by the set $B_{m,j}$, the
cells in the second will be indexed by subsets of $B_{m,j}$ of size r. A Schubert cell of
$G_{m,j}^r$ corresponding to a subset $S = \{(k_1, l_1), \ldots (k_r, l_r)\} \subset B_{m,j}$ will consist of those
$(r-1)$-dimensional projective subspaces of $P_{m,j}$ which intersect $V_{k_i, l_i, j}$ nontrivially
for all $1 \leq i \leq r$. We will denote the cell corresponding to S in $G_{m,j}^r$ by $V_{S,j}$. We
obtain a locally closed decomposition

$$G_{m,j}^r = \bigsqcup_{\substack{S \subseteq B_{m,j} \\ |S|=r}} V_{S,j}.$$

Occasionally, when it is clear from the context that S is a subset of $B_{m,j}$, we will
supress the index j and write just V_S for the Schubert cells of $G_{m,j}^r$.

We will call a Schubert cell *maximal* if it intersects the maximal dimensional
cell of $P_{m,j}$ nontrivially. Such a cell corresponds to subsets $S \subset B_{m,j}$ which contain
(k_{min}, l) where k_{min} is minimal among the first components of the elements of $B_{m,j}$.

The intersection with $V_{k_{min},l,j}$ of a subspace corresponding to a point in a maximal Schubert cell is an affine subspace of $V_{k_{min},l,j}$. Conversely, to any affine subspace of $V_{k_{min},l,j}$, there corresponds a point in a maximal Schubert cell given by the completion of the subspace in $P_{m,j}$.

For a maximal subset S, denote by $\overline{S} \subset B_{m,j}$ the set of indices which we get by deleting (k_{min}, l) from S. $\overline{S}$ is empty, if $|S| = 1$. Define the codimension one projective subspace

$$\overline{P}_{m,j} = \bigsqcup_{\{(k,l) \in B_{m,j}\,:\,k > k_{min}\}} V_{k,l,j} \subset P_{m,j} = P_{m,j} \setminus V_{k_{min},l,j}.$$

For each $(r-1)$-dimensional subspace $U \subset P_{m,j}$ intersecting the affine space $V_{k_{min},l,j}$ nontrivially, let $\overline{U} = U \cap \overline{P}_{m,j}$.

LEMMA 5.4. *The map* $\omega \colon V_{S,j} \to V_{\overline{S},j}$ *defined by* $\omega(U) = \overline{U}$ *is a trivial affine fibration with fibre* $\mathbb{A}^{|B_{m,j}|-|S|}$.

We can think of this map as associating to an *affine* subspace of $V_{k_{min},l,j}$ its set of "ideal points at infinity".

Consider the fibre $\omega^{-1}(\overline{U})$ over a point $\overline{U} \in V_{\overline{S}}$, which we will also denote by $V_S|_{\overline{U}}$ below. This fibre consists of those subspaces $U \subseteq P_{m,j}$ which intersect $\overline{P}_{m,j}$ in $\overline{U}$, i.e. when considered as an affine subspace of $V_{k_{min},l,j}$, they have $\overline{U}$ as their set of "points at infinity". We will denote the set of such subspaces also by $V_{k_{min},l,j}/\overline{U}$. This notation means that we take the cosets in $V_{k_{min},l,j}$ of an arbitrary affine subspace $U \subset V_{k_{min},l,j}$ with $\overline{U} = \overline{P}_{m,j} \cap U$. The affine structure on $V_{k_{min},l,j}$ descends to an affine structure on $V_{k_{min},l,j}/\overline{U}$ which does not depend on the particular affine subspace U whose cosets were taken.

We define the mapping $\omega \colon V_{S,j} \to V_{\overline{S},j}$ as the identity for those index sets and the corresponding cells which are not maximal. In such cases, $\overline{S} = S$ considered as a subset of $B_{m,j} \setminus \{(k_{min}, l)\}$.

We also need a description of the affine subspaces of the cells $V_{k,l,j}$ for $k > k_{min}$. The relevant Schubert cells in this case are indexed by those subsets S of $B_{m,j}$ which contain (k,l) but do not contain any (k',l') for $k' < k$. Hence, the index set $B_{m,j}$ is first truncated by deleting the pairs (k',l') with $k' < k$. We will denote the result as $B_{m,j}(k)$. Then, the maximal Schubert cells for $V_{k,l,j}$ correspond to those subsets $S \subseteq B_{m,j}(k)$ of the truncated index set which contain (k,l). For these, $\overline{S}$ is defined by removing (k,l) from S. There is still a morphism $\omega \colon V_{S,j} \to V_{\overline{S},j}$ which is defined in the same way as above; its global structure and the description of its fibres is analogous to the previous special case.

Note that the notation $\overline{S}$ is ambiguous at this point: any maximal subset $S \subseteq B_{m,j}(k)$ can also be considered as a nonmaximal subset of $B_{m,j}(k')$ for $k' < k$. If we view S as a subset of $B_{m,j}(k)$, then $\overline{S} = S \setminus \{(k, m-k)\}$. On the other hand, if we view it as a nonmaximal subset of $B_{m,j}(k')$ for $k' < k$, then $\overline{S} = S$. We have

decided not to introduce extra notation; when this notation gets used below, we will always specify the reference point k explicitly.

5.6. The Young wall associated with a homogeneous ideal

In this section, we study ideals generated by degree- and weight-homogeneous polynomials; we will call such ideals simply *homogeneous* ideals. Here is the main definition of this section.

DEFINITION 5.5. Consider a homogeneous G_Δ-invariant ideal $I \lhd \mathbb{C}[x,y]$. Let Y_I denote the following subset of the transformed Young wall pattern of type D_n: for each block or half-block (k,l) of label j, with $k + l = m$, include this block or half-block in Y_I if and only if $I \cap S_m[\rho_j]$ does not intersect the preimage in $S_m[\rho_j]$ of the stratum $V_{k,l,j} \subset P_{m,j}$ from the stratification of Proposition 5.1. Y_I will be called the *profile* of I.

It will be useful to introduce a little bit of extra notation, and to reformulate this definition in the new notation. Given a homogeneous G_Δ-invariant ideal I, let $I_{m,j}$ be the set of G_Δ-invariant subspaces of minimal dimension in $I \cap S_m[\rho_j]$. Then as $I \cap S_m[\rho_j] \subset S_m[\rho_j]$ is a linear subspace, $I_{m,j} \subset P_{m,j}$ is a projective linear subspace. Then the definition simply says that a block or half-block (k,l) labelled j is included in Y_I if and only if $I_{m,j} \cap V_{k,l,j} = \emptyset$, for $m = k + l$ as before. Since $\{V_{k,l,j}\}$ is a standard stratification of the projective space $P_{m,j}$ into affine spaces, $\{I_{m,j} \cap V_{k,l,j}\}$ is also a standard stratification of its projective linear subspace $I_{m,j}$ into affine spaces, and so has the same number of strata as its affine dimension. We conclude

LEMMA 5.6. *For all m, j, the number of absent blocks or half-blocks of label j on the m-th diagonal equals $\dim(I \cap S_m[\rho_j])$.*

PROPOSITION 5.7. *Given a homogeneous G_Δ-invariant ideal $I \lhd \mathbb{C}[x,y]$, the associated subset Y_I of the transformed Young wall pattern of type D_n has the following properties.*

(1) *If a full or half block is missing, then all the blocks above-right from it on the diagonal are missing.*[4]
(2) *If a full block is missing, then all full or half blocks to the right of it are missing, and* at least *one (full or half) block immediately above it is missing.*
(3) *If a half block is missing, then the full block to the right of it is missing.*
(4) *If both half-blocks sharing the same block position are missing, then the full block immediately above this position is missing.*

In particular, if I is of finite codimension, then Y_I is a Young wall of type D_n, an element of the set $\mathcal{Z}_\Delta$.

---

[4]Again, for a missing half block only the half blocks of the same orientation have to be missing.

REMARK 5.8. As it can be checked from the definitions, the relationship between the directions in the original and transformed Young wall patterns is the following: (right, up, diagonally right and down) in the original correspond after transformation to (diagonally right and up, right, up) respectively. This way, it is easy to check the correspondence between the rules for Young walls from 5.2 and this proposition.

PROOF OF PROPOSITION 5.7. Fix a homogeneous invariant ideal $I \lhd \mathbb{C}[x, y]$ and let Y_I be the corresponding subset of the Young wall pattern. Property (1) of Y_I follows by applying L_1, recalling the inductive nature of the stratification of $P_{m,j}$ using L_1. The inductice construction also implies that it suffices to check properties (2)-(4) for blocks missing on the bottom row.

Let us next prove (2) in the general case, when a full block in position $(0, l)$ in representation $j \in [3, n-3]$ is missing from Y_I; by the choice of j, both above and to the right of this block there are also full blocks. Since the block at $(0, l)$ is missing, there is an invariant 2-dimensional subspace $u \in I_{l,j} \cap V_{0,l,j}$ contained in I. Since u is in the lowest stratum $V_{0,l,j}$ of $P_{l,j}$, it has a basis one of whose members at least is not divisible by xy; without loss of generality, we may assume that this polynomial is $x^a p$ where a is a non-negative integer and p a polynomial in x, y not divisible by x, y. Now we can write

$$L_2 u = u_+ \oplus u_-$$

with $u_+ \in I_{l+1,j+1}$ and $u_- \in I_{l+1,j-1}$. Then u_+ must contain a polynomial with $x^{a+1} p$ as nonzero summand, so it cannot be in the image of L_1; so we have $u_+ \in V_{0,l+1,j+1}$. Similarly, u_- must contain a polynomial with $x^a y p$ as nonzero summand, so it cannot be in the image of L_1^2 and so $u_- \in V_{1,l,j-1}$. Thus indeed both the blocks in positions $(0, l+1)$ and $(1, l)$ are missing as claimed.

Let us now consider what changes if j is chosen such that there are half-blocks around. Suppose first that the half-blocks happen to lie to the right of our block labelled j. Then we have a decomposition

$$L_2 u = u_+^1 \oplus u_+^2 \oplus u_-,$$

with u_+^i both one-dimensional. In this case, it is easy to check that the polynomial $x^{a+1} p$ cannot itself generate a one-dimensional eigenspace, so both u_+^i will contain a polynomial with $x^{a+1} p$ as nonzero summand. Thus neither of these subspaces can be in the image of L_1, and so must lie in the large stratum. Hence both these blocks are missing.

Suppose now that the half-blocks happen to lie above our block labelled j. Then $L_2 u$ is either three- or four-dimensional. In the general case, it has dimension four and there is a decomposition

$$L_2 u = u_-^1 \oplus u_-^2 \oplus u_+$$

with u^i_- both one-dimensional. It special situations L_2u is only three dimensional, and one of the u^i_-'s is missing (see Lemma 6.24 below for a detailed analysis). In any case, $x^{a+1}p$ will lie in u_+, forcing that subspace to be in the large stratum. The other relevant polynomial x^ayp may or may not generate a one-dimensional invariant subspace, depending on the values of a, p; so at least one, possibly both, of u^1_-, u^2_- lies in the image of L_1 but not L^2_1, forcing them to lie in the corresponding stratum. So at least one, but possibly both, of the corresponding half-blocks must be missing. We remark here that the other u^i_-, if present, can be divisible by a higher power of L_1. This implies that in this case the ideal generated by u may have nontrivial intersection with the cells $V_{k,l,j}$ even with $l > 0$.

The proofs of (3)-(4) follow the same pattern; we omit the details. Finally if I is of finite codimension, then it contains S_m for m large enough, and so Y_I contains only finitely many blocks and half-blocks. $\qquad\square$

CHAPTER 6

Type D_n: decomposition of the orbifold Hilbert scheme

In this chapter we give a cell decomposition of the equivariant Hilbert scheme for the type D case. This is based on a detailed analysis of the geometry of some naturally defined incidence varieties. In some cases these incidence varieties happen to have the structure of a join of two lower dimensional varieties.

6.1. The decomposition

The aim of this chapter is to prove the following result, which gives a constructive proof of Theorem 3.2 for type D_n.

THEOREM 6.1. *Let G_Δ be the subgroup of $SL(2,\mathbb{C})$ of type D_n. Then there is a locally closed decomposition*

$$\mathrm{Hilb}([\mathbb{C}^2/G_\Delta]) = \bigsqcup_{Y \in \mathcal{Z}_\Delta} \mathrm{Hilb}([\mathbb{C}^2/G_\Delta])_Y$$

of the equivariant Hilbert scheme $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])$ into strata indexed bijectively by the set $\mathcal{Z}_\Delta$ of Young walls of type D_n, with each stratum $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])_Y$ a non-empty affine space.

PROOF. The affine plane $\mathbb{C}^2$ carries the diagonal $T = \mathbb{C}^*$-action, which commutes with the G_Δ-action. The action of T lifts to all the equivariant Hilbert schemes $\mathrm{Hilb}^\rho([\mathbb{C}^2/G_\Delta])$ which are themselves nonsingular. Thus the fixed point set

$$\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])^T = \sqcup_\rho \mathrm{Hilb}^\rho([\mathbb{C}^2/G_\Delta])^T$$

is also a union of nonsingular varieties, and it consists of points representing homogeneous invariant ideals. The construction of 5.6 associates a Young wall Y to each homogeneous invariant ideal $I \lhd \mathbb{C}[x,y]$. Since the construction uses a locally closed decomposition of the projective spaces $P_{m,j}$, the Young wall Y also depends in a locally closed way on the ideal I, and thus we obtain a decomposition

$$\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])^T = \bigsqcup_{Y \in \mathcal{Z}_\Delta} Z_Y$$

into reduced locally closed subvarieties, where $Z_Y \subset \mathrm{Hilb}([\mathbb{C}^2/G_\Delta])^T$ is the locus of homogeneous invariant ideals I with associated Young wall Y.

Let $\mathrm{Hilb}([\mathbb{C}^2/G])_Y \subset \mathrm{Hilb}([\mathbb{C}^2/G])$ denote the locus of ideals which flow to Z_Y under the action of the torus T. Then by the Białynicki-Birula theorem [4], there is a regular map $\mathrm{Hilb}([\mathbb{C}^2/G])_Y \to Z_Y$ which is a Zariski locally trivial fibration with

affine space fibres, and a compatible T-action on the fibres. By Theorem 6.3 below, the base is an affine space as well. Hence, by [4, Sect.3, Remarks], $\mathrm{Hilb}([\mathbb{C}^2/G])_Y$ is an algebraic vector bundle over this base, and hence trivial (Serre–Quillen–Suslin) [42]. Theorem 6.1 follows.   □

REMARK 6.2.
  (1) As $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta]) = \mathrm{Hilb}(\mathbb{C}^2)^{G_\Delta} \subset \mathrm{Hilb}(\mathbb{C}^2)$ is a smooth subvariety, the universal family over $\mathrm{Hilb}(\mathbb{C}^2)$ restricts to a universal family over the equivariant Hilbert scheme $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])$. This restricts to a universal family of homogeneous invariant ideals $\mathcal{U} \lhd \mathcal{O}_{\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])^T} \otimes \mathbb{C}[x,y]$ over the T-fixed point set. Restricting this universal family $\mathcal{U}$ to each of the strata constructed above gives flat families of homogeneous invariant ideals $\mathcal{U}_Y \lhd \mathcal{O}_{Z_Y} \otimes \mathbb{C}[x,y]$ over each stratum Z_Y. It follows from the construction that the families $\mathcal{U}_Y$ are universal for flat families of homogeneous invariant ideals with associated Young wall Y. We will have occasion to use the universal property of the strata Z_Y below.
  (2) The diagonal $T = \mathbb{C}^*$-action on $\mathbb{C}^2$ induces the usual monomial grading on $\mathbb{C}[x,y]$, and hence on each quotient $\mathbb{C}[x,y]/I$ for I homogeneous. For an invariant ideal I this quotient carries a G_Δ representation. Hence we can define its multigraded Hilbert function. This means that there is a $\mathbb{Z}$-grading induced by the $T = \mathbb{C}^*$-action and a grading (or rather labelling) by the set $\{\rho_0, \ldots, \rho_n\}$ associated to the G_Δ-action. By Lemma 5.6, the multigraded Hilbert function of a homogeneous invariant ideal $I \lhd \mathbb{C}[x,y]$ is determined by its associated Young wall Y.

The following is the main technical result of this section.

THEOREM 6.3. *For each $Y \in \mathcal{Z}_\Delta$, the stratum Z_Y constructed above is isomorphic to a nonempty affine space.*

REMARK 6.4. We note that our proof of Theorem 6.3 below certainly provides some information about the dimension of the affine space Z_Y, and thus of the affine space $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])_Y$. We leave the study of these quantities, which could lead to a refinement of Theorem 3.1 in the Grothendieck ring of varieties for type D_n, for further study.

Our proof of Theorem 6.3, discussed below following some preparation, is a direct inductive proof. We start with a series of examples which exhibit the range of issues our proof will have to tackle; the discussions use results to be proved further below. Throughout we take the simplest example $n = 4$, which exhibits all the nontrivial features.

EXAMPLE 6.5. Let Y_1 be the triangle of size 3.

		0
	1	2
0	2	3 / 4
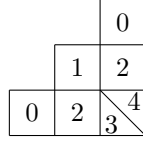
An invariant homogeneous ideal I corresponding to this Young wall necessarily has a generator in $V_{0,3,2} \subset P_{3,2}$. The latter is a projective line whose points can be represented by expressions $\alpha_0 L_2 L_3 + \alpha_1 L_2 L_4$. The affine line $V_{0,3,2}$ is given by $\alpha_0 + \alpha_1 \neq 0$. It is straightforward to check that a general point in $V_{0,3,2}$, that is, when $[\alpha_0 : \alpha_1] \notin \{[1:0], [0:1]\}$, indeed generates an ideal I with Young wall Y_1. However, when α_0 or α_1 become zero, then I does not intersect $V_{1,3,4}$, respectively $V_{1,3,3}$, even though it should, so we have to add another generator to I from within the corresponding cell (see Proposition 6.26(5) below). Both these cells are points, so there is no further choice to make and thus the space Z_{Y_1} is isomorphic to an affine line. This example already illustrates the fact that even within a single stratum Z_Y, the minimal number of generators of an ideal I with Young wall Y can vary.

EXAMPLE 6.6. Let Y_2 be the triangle of size 4. In this case, we get $Z_{Y_2} \cong Z_{Y_1} \cong \mathbb{A}^1$, the affine line of Example 6.5, due to the isomorphism $V_{0,3,2} \cong V_{0,4,0} \times V_{0,4,1}$, see Proposition 6.26(2) below, or repeat the same argument as above.

EXAMPLE 6.7. Let Y_3 be the triangle of size 5.

				0
			1	2
		0	2	3 / 4
	1	2	4 / 3	2
0	2	3 / 4	2	1 / 0
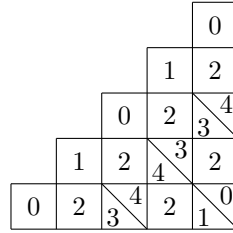
For each fixed ideal I with associated Young wall Y_3 must necessarily have an generator $f \in I$ such that $[f] \in V_{0,5,2}$. By construction, $V_{0,5,2}$ is an affine plane. This generator is, up to scalar, unique, since the block of label 2 in position $(0,5)$ is the only one with this label missing from the degree 5 antidiagonal. In other words, $I_{5,2} \cap V_{0,5,2}$ should be a point, otherwise the points at infinity of $I_{5,2} \cap V_{0,5,2}$ would intersect the other cells of degree 5 which is not allowed because of the shape of the Young wall. I must also intersect both $V_{1,5,0}$ and $V_{1,5,1}$, and again in essentially unique points, the corresponding blocks being the only 0/1 blocks missing from the degree 6 antidiagonal. This puts the following constraint on the allowed $[f] \in V_{0,5,2}$. Use the isomorphism L_1^{-1} which maps $V_{1,5,0} \sqcup V_{1,5,1}$, a disjoint union of a point and an affine line, to $V_{0,4,1} \sqcup V_{0,4,0}$. Map this locus into $V_{0,5,2}$, an affine plane, to obtain $M_1^0 \sqcup M_0^0 \subset V_{0,5,2}$ by taking the ideals generated by them (the curious notation $M_1^0 \sqcup M_0^0$ for this locus is used here to be consistent with the general setup later, see the definitions after Lemma 6.27). Then by Proposition 6.26(2) below, the ideal $\langle f \rangle$ intersects $P_{6,0}$ and $P_{6,1}$ at most in the correct cells $V_{1,5,0}$ and $V_{1,5,1}$ if and only

if $[f] \in V_{0,5,2}$ lies on the linear join of the affine line M_0^0 and the point M_1^0 inside the affine plane $V_{0,5,2}$ but not on $M_1^0 \sqcup M_0^0$. This join is the plane $V_{0,5,2}$ minus a punctured affine line $M_1 \setminus M_1^0$, where M_1 is the line parallel to M_0^0, going through M_1^0. On this join, but away from the locus $M_0^0 \sqcup M_1^0$ itself, we can set $I = \langle f \rangle$ to indeed get an ideal with Young wall Y_3. On the locus $M_0^0 \sqcup M_1^0$ however, the ideal $\langle f \rangle$ itself will not actually meet both cells, so we have to add an arbitrary generator of the missed cell to f to obtain an ideal of the correct Young wall. Over the affine line M_0^0, there is no choice, since $V_{1,5,0}$ is a point. But over the point M_1^0, we still have $V_{1,5,0}$, in other words an affine line, worth of choices. This sofar tells us that Z_{Y_3} is the disjoint union of $V_{0,5,2} \setminus M_1 \cong \mathbb{A}^2 \setminus \mathbb{A}^1$ and $V_{1,5,0} \cong \mathbb{A}^1$.
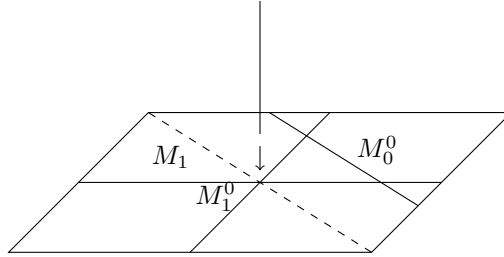
To fully work out the geometry of Z_{Y_3}, note that $P_{5,2} \cong \mathbb{P}^2$ can be parameterized by expersions $\alpha_0 L_2 L_3^2 + \alpha_1 L_2 L_3 L_4 + \alpha_2 L_2 L_4^2$. The locus $V_{0,5,2} \subset P_{5,2}$ is given by $\alpha_0 + \alpha_1 + \alpha_2 \neq 0$. The image of $V_{0,4,0}$ in these coordinates is $M_0^0 = \{(\alpha_0, 0, \alpha_1) : \alpha_0 + \alpha_1 \neq 0\}$, while the image of $V_{0,4,1}$ is $M_1^0 = \{(0, 1, 0)\}$. The linear combinations of the points in M_0^0 and M_1^0 cover the whole affine plane $V_{0,5,2}$, except a punctured line. For a general linear combination $a \cdot (\alpha_0, 0, \alpha_1) + b \cdot (0, 1, 0)$, the ideal generated by the corresponding f intersects $V_{1,5,0} \times V_{1,5,1}$ in $(L_1 L_3 L_4, \alpha_0 L_1 L_3^2 + \alpha_1 L_1 L_4^2)$. For $(a, b) = (1, 0)$ we have to have an extra generator in $V_{1,5,0}$, while for $(a, b) = (0, 1)$ we need an extra generator in $V_{1,5,1}$.

Consider a family of ideals which approaches the point M_1^0 from the direction $(\alpha_0, 0, \alpha_1)$. Then it can be shown by explicit calculation that the limit ideal contains the subspace generated by $\alpha_0 L_1 L_3^2 + \alpha_1 L_1 L_4^2 \in V_{1,5,1}$. This shows that Z_{Y_3} can be obtained by blowing up the affine plane $V_{0,5,2}$ in its point M_1^0, and removing the proper transform of the punctured line $M_1 \setminus M_1^0$ from this blowup. Thus $Z_{Y_3} \cong \mathbb{A}^2$. From the blowup construction, we also obtain a canonical morphism $Z_{Y_3} \to \mathbb{A}^1$, the restriction of the morphism $\text{Bl}_0 \mathbb{A}^2 \to \mathbb{P}^1$ to the exceptional curve of the blowup.

EXAMPLE 6.8. Let Y_3' be the Young wall
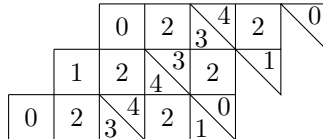
There is necessarily still a unique generator in $V_{0,5,2}$. The difference compared to Y_3 is that there is now no intersection with $V_{1,5,1}$ but an intersection with $V_{3,3,1}$. The cells of the 0/1-blocks missing from the degree six diagonal are $V_{1,5,0}$ and $V_{3,3,1}$, both

of which are zero dimensional. As before, we pull back these using L_1^{-1}, take the linear combinations of their images in $P_{5,2}$, and intersect this line with $V_{0,5,2}$. This is exactly the line M_1 from Example 6.7. This has one special point, M_1^0. If the new generator is (in the subspace represented by) this point, then it will not generate an ideal with shape Y, except if we keep the unique element of $V_{3,3,0}$ as a generator. In any case, $Z_{Y_3'} \cong M_1 \cong \mathbb{A}^1$.

EXAMPLE 6.9. It is well known that the minimal resolution of the singularity $\mathbb{C}^2/G_\Delta$ is given by the component $\mathrm{Hilb}^{\rho_{\mathrm{reg}}}([\mathbb{C}^2/G])$ of $\mathrm{Hilb}([\mathbb{C}^2/G])$ corresponding to the regular representation [40]. The $\mathbb{C}^*$-fixed set on the minimal resolution consists of the $\mathbb{P}^1$, the exceptional locus corresponding to the central node in the Dynkin diagram, as well as three isolated points on the other three $\mathbb{P}^1$'s. The following five Young walls contribute for the regular representation.
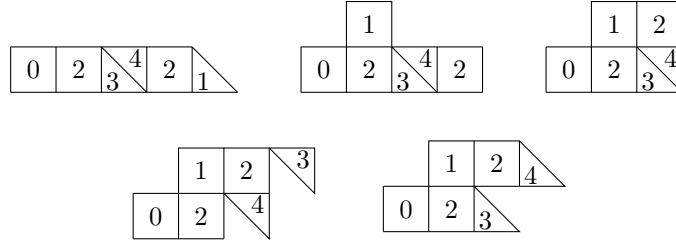
A quick computation shows that Z_Y is a point in each case, except for the last Young wall in the first row, when it is an affine line similarly as in Example 6.5 but in this case there is also a generator in $V_{2,2,0}$. This affine line contains the point corresponding to the Young wall next to it in its closure, giving the central $\mathbb{P}^1$.

6.2. Incidence varieties

The purpose of this section is to introduce some incidence varieties inside products of the Schubert cells defined in 5.5. We state some propositions regarding these incidence varieties and morphisms between them, whose proofs we defer to 6.6 below. We discuss four different cases.

Case 6.2.1 Assume that $m \equiv 0 \bmod n - 2$ is a nonnegative integer, such that at position $(0, m)$ there is a divided block with labels (c_1, c_2). Let c be the label of the block at position $(1, m)$. Let $S_c \subseteq B_{m+1,c}$ be a nonempty maximal subset. Let $S_1 \subseteq B_{m,c_1}$ and $S_2 \subseteq B_{m,c_2}$ be two maximal subsets which are *allowed* by S_c. This means, by definition, that each block above every composite block whose two halves are both contained in $S_1 \cup S_2$, and each block to the right of every composite block at least one half-block of which in $S_1 \cup S_2$, is in S_c.

Consider the incidence varieties

$$X_{S_1,S_2}^{S_c} = \{(U_1, U_2, U_c) \ : \ (U_1, U_2) \cap P_{m+1,c} \subseteq U_c\} \subseteq V_{S_1} \times V_{S_2} \times V_{S_c,c},$$

and

$$Y_{\overline{S}_1,\overline{S}_2}^{S_c} = \{(\overline{U}_1, \overline{U}_2, U_c) \ : \ (\overline{U}_1, \overline{U}_2) \cap P_{m+1,c} \subseteq U_c\} \subseteq V_{\overline{S}_1} \times V_{\overline{S}_2} \times V_{S_c,c}.$$

These varieties fit into the diagram

$$
\begin{array}{ccccc}
X^{S_c}_{S_1,S_2} & \subseteq & V_{S_1} \times V_{S_2} \times V_{S_c,c} & \xrightarrow{\ \mathrm{Id}\times\mathrm{Id}\times\omega\ } & V_{S_1} \times V_{S_2} \times V_{\overline{S}_c,c} \\
 & & \downarrow{\scriptstyle \omega\times\omega\times\mathrm{Id}} & & \downarrow{\scriptstyle \omega\times\omega\times\mathrm{Id}} \\
Y^{S_c}_{\overline{S}_1,\overline{S}_2} & \subseteq & V_{\overline{S}_1} \times V_{\overline{S}_2} \times V_{S_c,c} & \xrightarrow{\ \mathrm{Id}\times\mathrm{Id}\times\omega\ } & V_{\overline{S}_1} \times V_{\overline{S}_2} \times V_{\overline{S}_c,c}\ .
\end{array}
$$

PROPOSITION 6.10.

(1) The image of $X^{S_c}_{S_1,S_2}$ under the vertical morphism $V_{S_1} \times V_{S_2} \times V_{S_c,c} \to V_{\overline{S}_1} \times V_{\overline{S}_2} \times V_{S_c,c}$ is precisely $Y^{S_c}_{\overline{S}_1,\overline{S}_2}$.

(2) The induced morphism $X^{S_c}_{S_1,S_2} \to Y^{S_c}_{\overline{S}_1,\overline{S}_2}$ is a trivial fibration over its image with affine fibers of dimension $|S_c| - |S_1| - |S_2| + 1$.

(3) The horizontal morphism $X^{S_c}_{S_1,S_2} \to V_{S_1} \times V_{S_2} \times V_{\overline{S}_c,c}$ is injective.

Case 6.2.2 Let $m \equiv 0 \bmod n-2$, but this time consider only one half block of label $c_0 = \kappa(c)$ at the position $(0,m)$. Let $S_c \subseteq B_{m+2,c}$ be a nonempty maximal subset, and $S \subseteq B_{m,\kappa(c)}$ be a maximal subset which is allowed by S_c. This means that for each block in S there is a block in S_c at the top right corner. In analogy with the previous case, let

$$
X^{S_c}_{S} = \{(U,U_c)\ :\ (U) \cap P_{m+2,c} \subseteq U_c\} \subseteq V_S \times V_{S_c,c}\ ,
$$

and

$$
Y^{S_c}_{\overline{S}} = \{(\overline{U},U_c)\ :\ (\overline{U}) \cap P_{m+2,c} \subseteq U_c\} \subseteq V_{\overline{S}} \times V_{S_c,c}\ ,
$$

which fit into the diagram

$$
\begin{array}{ccccc}
X^{S_c}_{S} & \subseteq & V_S \times V_{S_c,c} & \xrightarrow{\ \mathrm{Id}\times\omega\ } & V_S \times V_{\overline{S}_c,c} \\
 & & \downarrow{\scriptstyle \omega\times\mathrm{Id}} & & \downarrow{\scriptstyle \omega\times\mathrm{Id}} \\
Y^{S_c}_{\overline{S}} & \subseteq & V_{\overline{S}} \times V_{S_c,c} & \xrightarrow{\ \mathrm{Id}\times\omega\ } & V_{\overline{S}} \times V_{\overline{S}_c,c}\ .
\end{array}
$$

PROPOSITION 6.11.

(1) The image of $X^{S_c}_{S}$ under the vertical morphism $V_S \times V_{S_c,c} \to V_{\overline{S}} \times V_{S_c,c}$ is exactly $Y^{S_c}_{\overline{S}}$.

(2) The induced morphism $X^{S_c}_{S} \to Y^{S_c}_{\overline{S}}$ is a trivial fibration over its image with affine fibers of dimension $|S_c| - |S|$.

(3) The horizontal morphism $X^{S_c}_{S} \to V_S \times V_{\overline{S}_c,c}$ is injective.

Case 6.2.3 Let $m \equiv 1 \bmod n-2$, and c_1 and c_2 the labels of the divided block immediately above the block at position $(0,m)$. Let $S_1 \subseteq B_{m+1,c_1}$, $S_2 \subseteq B_{m+1,c_2}$ be nonempty subsets at least one of which is maximal. Let moreover, $S \subseteq B_{m,j}$ be a maximal subset which is allowed by S_1 and S_2. In this case, this means the following: for each block b in S, there is a divided block of with labels (c_1,c_2) in the pattern either immediately above or to the right of b. In the first case, we require that at

least one of these half-blocks is in $S_1 \cup S_2$. In the second case, we require that both are contained in $S_1 \cup S_2$.

Given this data, we define

$$X_S^{S_1,S_2} = \{(U, U_1, U_2) \ : \ (U) \cap P_{m+1,c_1} \subseteq U_1, (U) \cap P_{m+1,c_2} \subseteq U_2\} \subseteq V_S \times V_{S_1,c_1} \times V_{S_2,c_2} \ ,$$

and

$$Y_{\overline{S}}^{S_1,S_2} = \{(\overline{U}, U_1, U_2) \ : \ (\overline{U}) \cap P_{m+1,c_1} \subseteq U_1, (\overline{U}) \cap P_{m+1,c_2} \subseteq U_2\} \subseteq V_{\overline{S}} \times V_{S_1,c_1} \times V_{S_2,c_2} \ .$$

We now have the following diagram:

$$
\begin{array}{ccccc}
X_S^{S_1,S_2} & \subseteq & V_S \times V_{S_1,c_1} \times V_{S_2,c_2} & \xrightarrow{\ \mathrm{Id} \times \omega \times \omega\ } & V_S \times V_{\overline{S}_1,c_1} \times V_{\overline{S}_2,c_2} \\
& & \downarrow{\scriptstyle \omega \times \mathrm{Id} \times \mathrm{Id}} & & \downarrow{\scriptstyle \omega \times \mathrm{Id} \times \mathrm{Id}} \\
Y_{\overline{S}}^{S_1,S_2} & \subseteq & V_{\overline{S}} \times V_{S_1,c_1} \times V_{S_2,c_2} & \xrightarrow{\ \mathrm{Id} \times \omega \times \omega\ } & V_{\overline{S}} \times V_{\overline{S}_1,c_1} \times V_{\overline{S}_2,c_2} \ .
\end{array}
$$

PROPOSITION 6.12.

(1) The image of $X_S^{S_1,S_2}$ under the vertical morphism $V_S \times V_{S_1,c_1} \times V_{S_2,c_2} \to V_{\overline{S}} \times V_{S_1,c_1} \times V_{S_2,c_2}$ is exactly $Y_{\overline{S}}^{S_1,S_2}$.

(2) The induced morphism $X_S^{S_1,S_2} \to Y_{\overline{S}}^{S_1,S_2}$ is a trivial fibration with fibers isomorphic to affine spaces of dimension $|S_1| + |S_2| - |S|$.

We remark that the analogue of (3) of Propositions 6.10 and 6.11 is not true in this case. What happens to $X_S^{S_1,S_2}$ when we project $V_{S_1,c_1} \times V_{S_2,c_2}$ to $V_{\overline{S}_1,c_1} \times V_{\overline{S}_2,c_2}$ will be the subject of 7 below.

Case 6.2.4 Finally, assume $m \not\equiv 0, 1 \bmod n - 2$ with a full block in position $(0, m)$. Let c be the label of the full block immediately above this position, and $S_c \subseteq B_{m+1,c}$ a nonempty maximal subset. Let moreover $S \subseteq B_{m,j}$ be a maximal subset which is allowed by S_c; in this case, this means that above every block of S there is a block in S_c. Consider the incidence varieties

$$X_S^{S_c} = \{(U, U_c) \ : \ (U) \cap P_{m+1,c} \subseteq U_c\} \subseteq V_S \times V_{S_c,c}$$

and

$$Y_{\overline{S}}^{S_c} = \{(\overline{U}, U_c) \ : \ (\overline{U}) \cap P_{m+1,c} \subseteq U_c\} \subseteq V_{\overline{S}} \times V_{S_c,c}.$$

There is the following diagram:

$$
\begin{array}{ccccc}
X_S^{S_c} & \subseteq & V_S \times V_{S_c,c} & \xrightarrow{\ \mathrm{Id} \times \omega\ } & V_S \times V_{\overline{S}_c,c} \\
& & \downarrow{\scriptstyle \omega \times \mathrm{Id}} & & \downarrow{\scriptstyle \omega \times \mathrm{Id}} \\
Y_{\overline{S}}^{S_c} & \subseteq & V_{\overline{S}} \times V_{S_c,c} & \xrightarrow{\ \mathrm{Id} \times \omega\ } & V_{\overline{S}} \times V_{\overline{S}_c,c} \ .
\end{array}
$$

PROPOSITION 6.13.

(1) The image of $X_S^{S_c}$ under the vertical morphism $V_S \times V_{S_c,c} \to V_{\overline{S}} \times V_{S_c,c}$ is exactly $Y_{\overline{S}}^{S_c}$.

(2) *The induced morphism* $X_S^{S_c} \to Y_{\overline{S}}^{S_c}$ *is a trivial fibration over its image with affine fibers of dimension* $|S_c| - |S|$.

(3) *The horizontal morphism* $X_S^{S_c} \to V_S \times V_{\overline{S}_{c,c}}$ *is injective.*

6.3. Proof of Theorem 6.3

In this section we prove Theorem 6.3, thus completing the proof of Theorem 6.1, using the constructions and results stated in 6.2. Given a Young wall $Y \in \mathcal{Z}_\Delta$, we need to show that the corresponding stratum Z_Y is an affine space. The following is the key combinatorial definition which underlies much of the rest of the paper.

DEFINITION 6.14. Consider the Young wall Y, as usual in the transformed pattern. The *salient blocks* of Y are those blocks in the complement of Y, whose absence from Y does not follow from the shape of the rows below it, and which are at the leftmost positions in their rows with this property. In particular, these are

- missing half blocks under which there is a block in Y;
- missing undivided full blocks under which there is a block in Y;
- missing divided full blocks immediately to the right of the boundary of Y;
- the leftmost missing block(s) in the bottom row.

Given an ideal $I \in Z_Y$, it is easy to see I is necessarily generated by elements lying in cells corresponding to the salient blocks of Y. In most cases it is also true that all cells corresponding to salient blocks must contain a generator, but not always; we have already seen Example 6.5, where the divided missing blocks at position $(1,3)$ are salient blocks of Y_3, since they lie immediately to the right of the boundary of Y_3, but the corresponding cells do not necessarily contain generators of an ideal $I \in Z_{Y_3}$.

We start our analysis by defining maps from the strata Z_Y to the Grassmannian cells defined in 5.5. Consider an arbitrary block or half-block of label j at position (k,l) in the Young wall pattern. Let $S(k,l,j) \subseteq B_{k+l,j}$ be the set of blocks of label j at the positions (k',l') where $k + l = k' + l'$, $k' \geq k$, and which are not in Y. $S(k,l,j)$ is called the *index set* of (k,l) in Y. If the block of label j at position (k,l) is not contained in Y, then the index set $S = S(k,l,j)$ contains (k,l) as well. By the correspondence discussed at the end of Section 5.5 between the maximal Schubert cells of the relevant Grassmannian for $V_{k,l,j}$ and subsets of $B_{k+l,j}(k)$ which do contain (k,l), there is a maximal Schubert cell $V_{S,j}$ corresponding to S. The affine cell $V_{S,j} \subset G_{k+l,j}^r$ for $r = |S|$ parameterizes certain affine subspaces of $V_{k,l,j}$, or, equivalently, projective subspaces of $P_{k+l,j}$, the projectivization of the space of degree $(k+l)$ homogeneous polynomials which transform in the representation ρ_j with respect to G_Δ. The correspondence is by projective closure of the corresponding affine subspace of $V_{k,l,j}$.

LEMMA 6.15. *For any block or half-block at position (k, l) which is not contained in Y and has index set $S = S(k, l, j)$, there is a morphism*

$$
\begin{aligned}
Z_Y &\to V_{S,j} \\
I &\mapsto I \cap V_{k,l,j}.
\end{aligned}
$$

PROOF. Assume first for simplicity that $(k, l) = (0, m)$. Let $\mathcal{U}_Y \lhd (\mathcal{O}_{Z_Y} \otimes \mathbb{C}[x, y])$ be the universal family of homogeneous ideals over Z_Y introduced in Remark 6.2. Consider the subfamily $\mathcal{V} = \mathcal{U}_Y \cap (\mathcal{O}_{Z_Y} \otimes S_m[\rho_j])$. This is a family of subspaces of dimension $r \dim \rho_j$ in $S_m[\rho_j]$ parameterized by Z_Y. It is know, that the morphisms from Z_Y to the equivariant Grassmannian $G_{m,j}^r$ of $S_m[\rho_j]$ are in one-to-one correspondence with G_Δ invariant subbundles of $\mathcal{O}_{Z_Y} \otimes S_m[\rho_j]$ of rank $r \dim \rho_j$ [13, p. 88, Theorem 2.4]. Hence, there is a classifying morphism $Z_Y \to G_{m,j}^r$ inducing $\mathcal{V}$.

By the definition of Z_Y, the multigraded Hilbert polynomial of $\mathcal{U}_Y$ is constant. The Hilbert polynomial encodes the dimensions of the intersections with the cells of $P_{m,j}$. Therefore, over closed points of Z_Y the elements of the family $\mathcal{V}$, when considered as projective subspaces of $P_{m,j}$, intersect exactly the cells $V_{k_i,l_i,j}$ for each element $(k_i, l_i) \in S$. According to Section 5.5, these projective subspaces are represented by points in the Schubert cell $V_{S,j} \subset G_{m,j}^r$. Hence, the image of the classifying morphism of $\mathcal{V}$ is inside the cell $V_{S,j}$. By the construction, the classifying morphism is just the same as taking the intersection of $I_{m,j}$ with $V_{0,m,j}$. This is denoted as $I \cap V_{0,m,j}$ above.

The general case follows similarly as in Section 5.5. □

We will prove Theorem 6.1 by induction on the number of nonempty rows of Y. Consider an arbitrary Young wall Y consisting of $l > 0$ rows. Let $\overline{Y}$ denote the Young wall obtained from Y by deleting its bottom row; we will call this the *truncation* of Y. Of course the labels of the half blocks are exchanged by κ, but we will suppress this in the notations. The following result will be key to our induction.

LEMMA 6.16. *There exists a morphism of schemes*

$$
\begin{aligned}
T : Z_Y &\to Z_{\overline{Y}} \\
I &\mapsto L_1^{-1} \left(I \cap L_1 \mathbb{C}[x, y] \right).
\end{aligned}
$$

PROOF. Let $\mathcal{U}_Y \lhd (\mathcal{O}_{Z_Y} \otimes \mathbb{C}[x, y])$ be the universal family of homogeneous ideals over Z_Y. Consider $\mathcal{I} = L_1^{-1} (\mathcal{U}_Y \cap L_1(\mathcal{O}_{Z_Y} \otimes \mathbb{C}[x, y]))$. Is is straightforward to check locally that this is still a sheaf of ideals in $\mathcal{O}_{Z_Y} \otimes \mathbb{C}[x, y]$. On closed points of Z_Y, it is also clear that the restriction has Young wall $\overline{Y}$. As mentioned above, the multigraded Hilbert polynomial of $\mathcal{U}_Y$ is constant. As Z_Y is reduced, it then follows from [31, Ch III. Thm. 9.9] that $\mathcal{I}$ is a flat family of homogeneous ideals with Young wall $\overline{Y}$ over Z_Y. By Remark 6.2 there is a classifying morphism $Z_Y \to Z_{\overline{Y}}$ for this family, which is exactly the morphism T. □

Next, we continue with an investigation of the combinatorics of the bottom two rows of our Young wall Y. The boundary of Y in the transformed pattern is divided by the blocks into horizontal, vertical and diagonal straight line segments. The first two lines in the bottom can be connected in the following six possible ways:
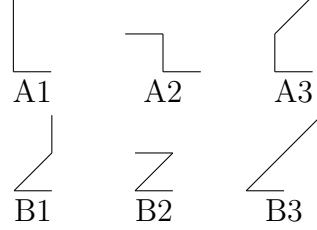
Here a diagonal straight line borders a half block of the Young wall, which can be either a lower or an upper triangle. In the A cases the salient block in the bottom row is a full block, while in the B cases it is a half block.

Let the salient block of Y in its bottom row be at position $(0, m)$. It can be either a divided or undivided full block, or a half block. In the first case, we have a type A corner at the bottom of Y, while in the second case there is a type B corner. As in 6.2, we need to consider four cases. In each case, we are going to define morphisms $Z_Y \to \mathcal{X}_Y$ and $Z_{\overline{Y}} \to \mathcal{Y}_{\overline{Y}}$ to incidence varieties defined in 6.2.

Case 6.3.1 Assume $m \equiv 0 \bmod n - 2$ and we have vertex types A1 or A2 (A3 is not possible in this case). We are in the context of Case 6.2.1: the divided block at position $(0, m)$ has labels (c_1, c_2), and index sets S_1, S_2; the block at position $(1, m)$ has label c, and index set S_c. By the Young wall rules for Y, S_1, S_2 is allowed by S_c. Lemma 6.15 implies that there is a morphism

$$\begin{aligned} Z_Y &\to & V_{S_1} \times V_{S_2} \times V_{S_c} \\ I &\mapsto & (I \cap V_{0,m,c_1}, I \cap V_{0,m,c_2}, I \cap V_{1,m,c}). \end{aligned}$$

By construction, the image of this morphism is contained in the incidence variety $X_{S_1,S_2}^{S_c} \subset V_{S_1} \times V_{S_2} \times V_{S_c}$ from Case 6.2.1. Denote $\mathcal{X}_Y = X_{S_1,S_2}^{S_c} \subseteq V_{S_1} \times V_{S_2} \times V_{S_c}$; we thus obtain an induced morphism $Z_Y \to \mathcal{X}_Y$.

There is also a morphism

$$\begin{aligned} Z_{\overline{Y}} &\to & V_{\overline{S}_1} \times V_{\overline{S}_2} \times V_{S_c} \\ I &\mapsto & (L_1 I \cap V_{k_1,l_1,c_1}, L_1 I \cap V_{k_2,l_2,c_2}, L_1 I \cap V_{1,m,c}), \end{aligned}$$

where (k_i, l_i) is the lowest block in $\overline{S}_i$ for $i = 1, 2$. We obtain an induced morphism $Z_{\overline{Y}} \to \mathcal{Y}_{\overline{Y}}$, where $\mathcal{Y}_{\overline{Y}} = Y_{\overline{S}_1,\overline{S}_2}^{S_c}$.

Case 6.3.2 Assume $m \equiv 0 \bmod n-2$ with vertex types B1 to B3. This is Case 6.2.2: the block at position $(0, m)$ has label $\kappa(c)$, and index set $S = S_{\kappa(c)}$; the block at position $(1, m + 1)$ has label c, and index set S_c. We consider the morphisms

$$\begin{aligned} Z_Y &\to & V_S \times V_{S_c} \\ I &\mapsto & (I \cap V_{0,m,\kappa(c)}, I \cap V_{1,m+1,c}) \, , \end{aligned}$$

and

$$\begin{array}{rcl} Z_{\overline{Y}} & \to & V_{\overline{S}} \times V_{S_c} \\ I & \mapsto & (L_1 I \cap V_{k,l,\kappa(c)}, L_1 I \cap V_{1,m+1,c}), \end{array}$$

where again (k,l) is the lowest block in $\overline{S}$. In this case we let $\mathcal{X}_Y = X_S^{S_c}$ and $\mathcal{Y}_{\overline{Y}} = Y_{\overline{S}}^{S_c}$. The images of the morphisms above are contained in these.

Case 6.3.3 Assume $m \equiv 1 \bmod n-2$ with vertex types A1, A2, A3. This is Case 6.2.3: c_1 and c_2 are the labels of the divided block immediately above the block at position $(0,m)$, $S_1 \subseteq B_{m+1,c_1}$, $S_2 \subseteq B_{m+1,c_2}$ are their index sets, both (in cases A1 and A2) or one (in case A3) of which is maximal; S is the index set of the block at position $(0,m)$ with label j. We get a morphism

$$\begin{array}{rcl} Z_Y & \to & V_S \times V_{S_1,c_1} \times V_{S_2,c_2} \\ I & \mapsto & (I \cap V_{0,m,j}, I \cap V_{1,m,c_1}, I \cap V_{1,m,c_2}) \end{array}$$

whose image is contained in $\mathcal{X}_Y = X_S^{S_1,S_2}$. In this way we obtain an induced morphism $Z_Y \to \mathcal{X}_{Y,\overline{Y}}$. Similarly, consider the morphism

$$\begin{array}{rcl} Z_{\overline{Y}} & \to & V_{\overline{S}} \times V_{S_1,c_1} \times V_{S_2,c_2} \\ I & \mapsto & (L_1 I \cap V_{k,l,j}, L_1 I \cap V_{1,m,c_1}, L_1 I \cap V_{1,m,c_2}), \end{array}$$

where (k,l) is the lowest block in $\overline{S}$. We obtain an induced morphism $Z_{\overline{Y}} \to \mathcal{Y}_{\overline{Y}}$, where $\mathcal{Y}_{\overline{Y}} = Y_{\overline{S}}^{S_1,S_2}$.

Case 6.3.4 Assume finally that $m \not\equiv 1 \bmod n-2$ with vertex types A1 or A2. This is Case 6.2.4: the full block at position $(0,m)$ has label j and index set S which is maximal; the full block at position $(1,m)$ has label c and index set S_c; S is allowed by S_c. We get a morphism

$$\begin{array}{rcl} Z_Y & \to & V_S \times V_{S_c} \\ I & \mapsto & (I \cap V_{0,m,j}, I \cap V_{1,m,c}) \end{array}$$

whose image is contained in $X_S^{S_c} \subseteq V_S \times V_{S_c}$, an incidence variety we denote by $\mathcal{X}_Y$ to obtain an induced morphism $Z_Y \to \mathcal{X}_{Y,\overline{Y}}$.

Second, let

$$\begin{array}{rcl} Z_{\overline{Y}} & \to & V_{\overline{S}} \times V_{S_c} \\ I & \mapsto & (L_1 I \cap V_{k,l,j}, L_1 I \cap V_{1,m,c}), \end{array}$$

where (k,l) is the lowest block in $\overline{S}$. By letting $\mathcal{Y}_{\overline{Y}} = Y_{\overline{S}}^{S_c}$ we obtain an induced morphism $Z_{\overline{Y}} \to \mathcal{Y}_{Y,\overline{Y}}$.

The last key step in our inductive proof is the following result, valid in all four cases above.

PROPOSITION 6.17. *The following is a scheme-theoretic fiber product diagram, with the right hand vertical map in each case given by the map induced by statement*

(1) of Propositions 6.10, 6.11 6.12 or 6.13 as appropriate.

$$(6.1) \qquad \begin{array}{ccc} Z_Y & \longrightarrow & \mathcal{X}_Y \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \omega \times \mathrm{Id}} \\ Z_{\overline{Y}} & \longrightarrow & \mathcal{Y}_{\overline{Y}}. \end{array}$$

PROOF. It is immediate from the definitions in the different cases that the diagram is commutative. We thus need to show that it is a fibre product. Let B be an arbitrary base scheme and let $f \colon B \to Z_{\overline{Y}}$ and $g \colon B \to \mathcal{X}_Y$ be morphisms; we need to show that these induce a unique morphism $B \to Z_Y$. We consider Case 6.3.1; the proof in the other cases is analogous. The map f corresponds to a flat family of ideals $\mathcal{I}_f \lhd \mathcal{O}_B \otimes \mathbb{C}[x,y]$ with Young wall $\overline{Y}$. The map g corresponds to a 3-tuple of families $\mathcal{U}_{1,g}, \mathcal{U}_{2,g}, \mathcal{U}_{c,g}$ of subspaces of $\mathbb{C}[x,y]$ over B. Given this data, consider the family of ideals

$$\mathcal{I}_{f,g} = (L_1 \mathcal{I}_f, \mathcal{U}_{1,g}, \mathcal{U}_{2,g}) \lhd \mathcal{O}_B \otimes \mathbb{C}[x,y]$$

over B. Here there parentheses mean the generated ideal as explained in Section 6.5 below. By the compatibility of (f,g) it is immediate that the Young wall of the corresponding ideals is Y. The classifying map of this family is the unique possible extension of (f,g) to a morphism $B \to Z_Y$. $\qquad \square$

CONCLUSION OF THE PROOF OF THEOREM 6.3. Assume that we have shown for any Young wall Y having less then l rows that the corresponding stratum Z_Y is affine, the $l = 1$ case being obvious. Consider an arbitrary Young wall Y consisting of l rows. Let $\overline{Y}$ denote its truncation, as defined above. By the induction assumption, the space $Z_{\overline{Y}}$ is affine. Also, by Propositions 6.10, 6.11, 6.12 or 6.13 respectively, the map $\mathcal{X}_Y \to \mathcal{Y}_{\overline{Y}}$ of Proposition 6.17 is a trivial affine fibration in all cases. By Proposition 6.17, the map $Z_Y \to Z_{\overline{Y}}$ is a pullback of a trivial affine fibration and thus itself a trivial affine fibration. Using the induction hypothesis, Z_Y is thus an affine space. The proof of Theorem 6.3 is complete. $\qquad \square$

REMARK 6.18. One can deduce from the above proof that one can in fact *canonically* choose generators of a homogeneous ideal $I \in Z_Y$, which are in the cells of the some of the salient blocks of Y; as discussed before, not all salient cells necessarily contain a generator. For describing the coarse Hilbert scheme we have to keep track of these generators, but we will do this only implicitly.

EXAMPLE 6.19. Returning to Examples 6.6-6.7, we see that for Y_3 the triangle of side 5, $\overline{Y_3} = Y_2$, the triangle of size 4. The map $T \colon Z_{Y_3} \cong \mathbb{A}^2 \to Z_{\overline{Y_3}} \cong \mathbb{A}^1$ is the map identified at the end of the discussion of Example 6.7.

6.4. Digression: join of varieties

We now make a digression to a necessary construction from algebraic geometry.

Recall [1] that the *join* $J(X, Y) \subset \mathbb{P}^n$ of two projective varieties $X, Y \subset \mathbb{P}^n$ is the locus of points on all lines joining a point of X to a point of Y in the ambient projective space. One well-known example of this construction is the following. Let $L_1 \cong \mathbb{P}^k$ and $L_2 \cong \mathbb{P}^{n-k-1}$ be two disjoint projective linear subspaces of $\mathbb{P}^n$.

LEMMA 6.20. *The join* $J(L_1, L_2) \subset \mathbb{P}^n$ *equals* $\mathbb{P}^n$. *Moreover, the locus* $\mathbb{P}^n \backslash (L_1 \cap L_2)$ *is covered by lines uniquely: for every* $p \in \mathbb{P}^n \setminus (L_1 \cap L_2)$, *there exists a unique line* $\mathbb{P}^1 \cong p_1 p_2 \subset \mathbb{P}^n$ *with* $p_i \in L_i$, *containing* p.

Let now $H \subset \mathbb{P}^n$ be a hyperplane not containing the L_i, which we think of as the hyperplane "at infinity". Let $V = \mathbb{P}^n \setminus H \cong \mathbb{A}^n$. Let $\overline{L}_i = L_i \cap H$, and let $L_i^o = L_i \setminus \overline{L}_i = L_i \cap V$ be the affine linear subspaces in V corresponding to L_i. Finally let $X = J(\overline{L}_1, L_2) \cong \mathbb{P}^{n-1}$ and $X^o = X \setminus (X \cap H) \cong \mathbb{A}^{n-1}$.

LEMMA 6.21. *Projection away from* L_1 *defines a morphism* $\phi \colon X \setminus L_1 \to L_2$, *which is an affine fibration with fibres isomorphic to* $\mathbb{A}^k$. ϕ *restricts to a morphism* $\phi^o \colon X^o \to L_2^o$, *which is a trivial affine fibration over* $L_2^o \cong \mathbb{A}^{n-k-1}$ *with the same fibers isomorphic to* $\mathbb{A}^k$.

In geometric terms, the map ϕ is defined on $X \setminus (L_1 \cup L_2)$ as follows: take $p \in X \setminus (L_1 \cup L_2)$, find the unique line $p_1 p_2$ passing through it, with $p_1 \in \overline{L}_1$ and $p_2 \in L_2$; then $\phi(p) = p_2$.

Let now U be a projective subspace of H which avoids $\overline{L}_2$. Let $U_1 \subset U$ be a codimension one linear subspace, and $W = U \setminus U_1$ its affine complement. In the main text, we need the following statement.

LEMMA 6.22. $\chi((J(L_2^o, W) \setminus L_2^o) \cap V) = 0$.

PROOF. With the same argument as in Lemma 6.21, $J(L_2^o, W) \cap V$ is a fibration over L_2^o with fiber $\mathrm{Cone}(W)$, and $(J(L_2^o, W) \setminus L_2^o) \cap V$ is a fibration over L_2^o with fiber $\mathrm{Cone}(W) \setminus \{\text{vertex}\}$. Since $\mathrm{Cone}(W) \setminus \{\text{vertex}\} = \mathbb{C}^* \times W$, the projection from $(J(L_2^o, W) \setminus L_2^o) \cap V \cong L_2^o \times W \times \mathbb{C}^*$ to $L_2^o \times W$ has fibers $\mathbb{C}^*$. The lemma follows. $\square$

The definition of join of varieties can be generalized for projective schemes $X, Y \subset \mathbb{P}^n_S$ over an arbitrary base scheme S [1]. With the same arguments it can be shown that this general definition satisfies too the properties mentioned above. Moreover, it satisfies the following base-change property as well:

LEMMA 6.23. *[1, B1.2] Let* S *be an arbitrary scheme. Then for schemes* $X, Y \subset \mathbb{P}^n_S$ *and an* S-*scheme* T, *we have the following equality in* $\mathbb{P}^n_T$:

$$J(X \times_S T, Y \times_S T) = J(X, Y) \times_S T.$$

6.5. Preparation for the proof of the incidence propositions

To prepare the ground for the proof of the propositions announced in 6.2, consider the operators defined in 5.4. We use these operators to describe projective

coordinates on some of the Grassmannians $P_{m,j}$. We first record the following equalities, computing the isotypical summands of the homogeneous pieces of the ring $S = \mathbb{C}[x, y]$.

LEMMA 6.24. *We have*

$$S_{2k(n-2)}[\rho_{\kappa(0,k)}] = (L_{n-1} + L_n)^{2k}[\rho_{\kappa(0,k)}] = (L_{n-1}^2 + L_n^2)^k;$$

$$S_{2k(n-2)}[\rho_{\kappa(1,k)}] = (L_{n-1} + L_n)^{2k}[\rho_{\kappa(1,k)}] = L_{n-1}L_n(L_{n-1}^2 + L_n^2)^{k-1};$$

$$S_{(2k+1)(n-2)}[\rho_{\kappa(n-1,k)}] = (L_{n-1} + L_n)^{2k+1}[\rho_{\kappa(n-1,k)}] = L_{n-1}(L_{n-1}^2 + L_n^2)^k;$$

$$S_{(2k+1)(n-2)}[\rho_{\kappa(n,k)}] = (L_{n-1} + L_n)^{2k+1}[\rho_{\kappa(n,k)}] = L_n(L_{n-1}^2 + L_n^2)^k.$$

PROOF. For each equality on the right, use (5.1) and an easy induction argument. For the equalities on the left, $\supseteq$ is always clear; then use dimension counting.   □

Thus, given an element $v \in P_{2k(n-2),\kappa(0,k)}$, we can write it uniquely in the form

$$v = \sum_{i=0}^{k} \alpha_i L_{n-1}^{2i} L_n^{2(k-i)},$$

for certain projective coordinates $[\alpha_0 : \cdots : \alpha_k]$. Analogous coordinates exist on $P_{2k(n-2),\kappa(1,k)}$ and $P_{(2k+1)(n-2),j}$ for $j = n - 1, n$.

For the two-dimensional representations we similarly have

LEMMA 6.25. *For $1 < j < n - 1$,*

$$
\begin{aligned}
S_{2k(n-2)+j-1}[\rho_j] &= L_j\left((L_{n-1} + L_n)^{2k}\right) \\
&= L_j\left((L_{n-1} + L_n)^{2k}[\rho_0]\right) \oplus L_j\left((L_{n-1} + L_n)^{2k}[\rho_1]\right);
\end{aligned}
$$

$$
\begin{aligned}
S_{(2k+2)(n-2)-j+1}[\rho_j] &= L_{n-m}\left((L_{n-1} + L_n)^{2k+1}\right) \\
&= L_{n-j}\left((L_{n-1} + L_n)^{2k+1}[\rho_{n-1}]\right) \oplus L_{n-j}\left((L_{n-1} + L_n)^{2k+1}[\rho_n]\right).
\end{aligned}
$$

PROOF. Analogous.   □

For such $1 < j < n-1$, the space $P_{2k(n-2)+j-1,j}$ of two-dimensional G_Δ-equivariant subspaces of $S_{2k(n-2)+j-1}[\rho_j]$ is a projective space as noted above. Using Lemma 6.25, we get a collection of distinguished two-dimensional G_Δ-equivariant subspaces $L_j L_{n-1}^i L_n^{2k-i}$ in $S_{2k(n-2)+j-1}[\rho_j]$; an arbitrary element $v \in P_{2k(n-2)+j-1,j}$ can be uniquely written as

$$v = \sum_{i=0}^{2k} \alpha_i L_j L_{n-1}^i L_n^{2k-i}$$

for certain projective coordinates $[\varepsilon_0 : \ldots : \varepsilon_{2k}]$. Analogous coordinates also exist on the space $P_{(2k+2)(n-2)-j+1,j}$.

For subspaces $U_1, \ldots, U_i \in \mathrm{Gr}(S)$ denote by $(U_1, \ldots, U_i)$ the G-invariant ideal of S generated by the corresponding subspaces. In particular, the ideal generated by (the subspaces represented by) points $v_1, \ldots, v_i \in \mathbb{P}S$ is denoted by $(v_1, \ldots, v_i)$. Then

$(U_1, \ldots, U_i)_{m,j}$ is represented by a projective linear subspace of $P_{m,j}$ (cf. Lemma 5.6). Thus, we can talk about its intersection with the cells of $P_{m,j}$. For simplicity we will use the notation $(U_1, \ldots, U_i) \cap V_{k,l,j} = (U_1, \ldots, U_i)_{k+l,j} \cap V_{k,l,j}$ for the intersection with $V_{k,l,j}$.

We need to study indidence relations between ideals generated by subspaces from the various strata defined above. First of all, let $v_j \in V_{0,m,j}$ for some m and j, corresponding to a full or half block. Denote by C the set of labels of full or half blocks immediately above or on the right of this block, i.e. in the positions $(1, m)$ or $(0, m + 1)$. Then $(v_j) \cap S_{m+1,c} = \emptyset$ whenever $c \notin C$. Indeed, one has to analyze the irreducible factors of $L_2 v_j$ as in the proof of Lemma 5.7. The following long statement discusses all the remaining cases when $c \in C$. It splits according to the different possibilities.

PROPOSITION 6.26.

(1) For $j = 0, 1$, fix $v_j \in V_{0,2k(n-2),j}$.
  (a) We have $(v_j) \cap \left(\bigcup_{l \geq 1} V_{l,2k(n-2)-l+1,2} \right) = \emptyset$. Hence the unique point $(v_j) \cap P_{2k(n-2)+1}$ necessarily lies in $V_{0,2k(n-2)+1,2}$. This provides an injection

$$V_{0,2k(n-2),j} \to V_{0,2k(n-2)+1,2}.$$

  (b) $(v_0, v_1) \cap \left(\bigcup_{l>1} V_{l,2k(n-2)-l+1,2} \right) = \emptyset$. In particular, the projective line $(v_0, v_1) \cap P_{2k(n-2)+1,2}$ necessarily intersects $V_{1,2k(n-2),2}$. Let the intersection point be $L_1 v_2$ for a certain $v_2 \in V_{0,2k(n-2)-1,2}$. Then v_2 is the unique point of $V_{0,2k(n-2)-1,2}$ such that $v_0, v_1 \in (v_2)$. As a consequence, for any projective subspace $U_2 \subseteq P_{2k(n-2)-1,2}$, the intersection $(v_0, v_1) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+1,2} \right)$ is contained in $L_1 U_2$ if and only if $v_0, v_1 \in (U_2)$.

(2) Let $v_2 \in V_{0,2k(n-2)+1,2}$. For $j = 0, 1$, if $(v_2) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+2,j} \right)$ is not empty, then it is necessarily one-dimensional, and it equals $L_1 v_j$ for a certain $v_j \in V_{0,2k(n-2),j}$. Exactly one of the following three cases happens.
  • $(v_2) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+2,1} \right) = L_1 v_0$ and $(v_2) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+2,0} \right) = \emptyset$. This happens if and only if $v_2 \in (v_0)$. In this case, $v_0 \in V_{0,2k(n-2),0}$, and $(v_2) \cap S_{2k(n-2)+2}$ has two (resp. three if $n = 4$) irreducible components: $L_1 v_0$ and $(v_2) \cap V_{0,2k(n-2)+2,3}$ (resp. $(v_2) \cap V_{0,2k(n-2)+2,3}$ and $(v_2) \cap V_{0,2k(n-2)+2,4}$).
  • $(v_2) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+2,1} \right) = \emptyset$ and $(v_2) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+2,0} \right) = L_1 v_1$ with symmetrical statements as in the previous case.
  • $(v_2) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+2,1} \right) = L_1 v_0$ and $(v_2) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+2,0} \right) = L_1 v_1$. This happens if and only if $v_2 \in (v_0, v_1)$ but $v_2 \notin (v_0) \cup (v_1)$. In this case at least one of the inclusions $v_0 \in V_{0,2k(n-2),0}$, $v_1 \in V_{0,2k(n-2),1}$ is satisfied but not necessarily both. Furthermore, $(v_2) \cap S_{2k(n-2)+2}$ has

three (resp. four if $n = 4$) irreducible components: $L_1 v_0$, $L_1 v_1$ and $(v_2) \cap V_{0,2k(n-2)+2,3}$ (resp. $(v_2) \cap V_{0,2k(n-2)+2,3}$ and $(v_2) \cap V_{0,2k(n-2)+2,4}$). Thus for $n > 4$, we obtain an isomorphism

$$V_{0,2k(n-2)+1,2} \quad \to \quad V_{0,2k(n-2)+2,3}$$
$$v_2 \quad \mapsto \quad (v_2) \cap V_{0,2k(n-2)+2,3},$$

whereas for $n = 4$, we get an isomorphism

$$V_{0,2k(n-2)+1,2} \quad \to \quad V_{0,2k(n-2)+2,3} \times V_{0,2k(n-2)+2,4}$$
$$v_2 \quad \mapsto \quad \left((v_2) \cap V_{0,2k(n-2)+2,3}, (v_2) \cap V_{0,2k(n-2)+2,4} \right).$$

Finally, for projective subspaces $U_0 \subseteq P_{2k(n-2),0}, U_1 \subseteq P_{2k(n-2),1}$,
- *the conditions $(v_2) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+2,1} \right) \subseteq L_1 U_0$ and $(v_2) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+2,0} \right) = \emptyset$ are satisfied if and only if $v_2 \in (U_0)$;*
- *the conditions $(v_2) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+2,1} \right) = \emptyset$ and $(v_2) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+2,0} \right) \subseteq L_1 U_1$ are satisfied if and only if $v_2 \in (U_1)$;*
- *the conditions $(v_2) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+2,1} \right) \subseteq L_1 U_0$ and $(v_2) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+2,0} \right) \subseteq L_1 U_1$ are satisfied if and only if $v_2 \in (U_0, U_1)$ but $v_2 \notin (U_0) \cup (U_1)$.*

(3) *Assume that $3 \leq j \leq n - 3$ (resp. $j = n - 2$) and set $v_j \in V_{0,2k(n-2)+j-1,j}$. Then $(v_j) \cap \left(\bigcup_{l>1} V_{l,2k(n-2)-l+j,j-1} \right) = \emptyset$. Furthermore, $(v_j) \cap S_{2k(n-2)+j}$ has two (resp. three) irreducible components: a point $(v_j) \cap V_{1,2k(n-2)+j-1,j-1}$ of the form $L_1 v_{j-1}$ for some $v_{j-1} \in V_{0,2k(n-2)+j-2,j-1}$, which is the unique element with $v_j \in (v_{j-1})$, and another point $(v_j) \cap V_{0,2k(n-2)+j,j+1}$ (resp. two other points $(v_j) \cap V_{0,2k(n-2)+j,n-1}$, $(v_j) \cap V_{0,2k(n-2)+j,n}$). We obtain an isomorphism*

$$V_{0,2k(n-2)+j-1,j} \to V_{0,2k(n-2)+j,j+1}$$

for $j \leq n - 3$ and an isomorphism

$$V_{0,2k(n-2)+j-1,j} \to V_{0,2k(n-2)+j,n-1} \times V_{0,2k(n-2)+j,n}$$

for $j = n - 2$. Also, for a projective subspace $U_{j-1} \subseteq P_{2k(n-2)+j-2,j-1}$, the intersection $(v_j) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+j,j-1} \right)$ is contained in $L_1 U_{j-1}$ if and only if $v_j \in (U_{j-1})$.

(4) *For $j = n - 1, n$, fix $v_j \in V_{0,(2k+1)(n-2),j}$.*
  (a) *We have $(v_j) \cap \left(\bigcup_{l\geq 1} V_{l,(2k+1)(n-2)-l+1,n-2} \right) = \emptyset$. Hence, the point $(v_j) \cap P_{(2k+1)(n-2)+1}$ is necessarily in $V_{0,(2k+1)(n-2)+1,n-2}$. This provides an injection*

$$V_{0,(2k+1)(n-2),j} \to V_{0,(2k+1)(n-2)+1,n-2}.$$

  (b) *$(v_{n-1}, v_n) \cap \left(\bigcup_{l>1} V_{l,(2k+1)(n-2)-l+1,2} \right) = \emptyset$. In particular, the projective line $(v_{n-1}, v_n) \cap P_{(2k+1)(n-2)+1,n-2}$ necessarily intersects $V_{1,(2k+1)(n-2),n-2}$. Let the intersection point be $L_1 v_{n-2}$ for a certain $v_{n-2} \in V_{0,(2k+1)(n-2)-1,n-2}$. Then v_{n-2} is the unique point of*

$V_{0,(2k+1)(n-2)-1,n-2}$ such that $v_{n-1}, v_n \in (v_{n-2})$. As a consequence, for any projective subspace $U_{n-2} \subseteq P_{(2k+1)(n-2)-1,n-2}$, the intersection $(v_{n-1}, v_n) \cap \left(\bigcup_{l>0} V_{l,(2k+1)(n-2)-l+1,n-2}\right)$ is contained in $L_1 U_{n-2}$ if and only if $v_{n-1}, v_n \in (U_{n-2})$.

(5) Let $v_{n-2} \in V_{0,(2k+1)(n-2)+1,n-2}$. For $j = n-1, n$ if $(v_{n-1}) \cap \left(\bigcup_{l>0} V_{l,(2k+1)(n-2)-l+2,j}\right) \neq \emptyset$ then this invariant subspace is one-dimensional, and it is of the form $L_1 v_j$ for certain $v_j \in V_{0,(2k+1)(n-2),j}$. Exactly one of the following three possibilities happens.

- $(v_{n-2}) \cap \left(\bigcup_{l>0} V_{l,(2k+1)(n-2)-l+2,n}\right) = L_1 v_{n-1}$ and $(v_{n-2}) \cap \left(\bigcup_{l>0} V_{l,(2k+1)(n-2)-l+2,n-1}\right) = \emptyset$. This happens if and only if $v_{n-2} \in (v_{n-1})$. In this case $v_{n-1} \in V_{0,(2k+1)(n-2),n-1}$, and $(v_2) \cap S_{(2k+1)(n-2)+2}$ has two (resp. three if $n = 4$) irreducible components: $L_1 v_{n-1}$ and $(v_{n-2}) \cap V_{0,(2k+1)(n-2)+2,n-3}$ (resp. $(v_2) \cap V_{0,(2k+1)(n-2)+2,0}$ and $(v_2) \cap V_{0,(2k+1)(n-2)+2,1}$).

- $(v_{n-2}) \cap \left(\bigcup_{l>0} V_{l,(2k+1)(n-2)-l+2,n}\right) = \emptyset$ and $(v_{n-2}) \cap \left(\bigcup_{l>0} V_{l,2k(n-2)-l+2,n-1}\right) = L_1 v_n$ with symmetrical statements as in the previous case.

- $(v_{n-2}) \cap \left(\bigcup_{l>0} V_{l,(2k+1)(n-2)-l+2,n}\right) = L_1 v_{n-1}$ and $(v_{n-2}) \cap \left(\bigcup_{l>0} V_{l,(2k+1)(n-2)-l+2,n-1}\right) = L_1 v_n$. This happens if and only if $v_{n-2} \in (v_{n-1}, v_n)$ but $v_{n-2} \notin (v_{n-1}) \cup (v_n)$. In this case at least one of the inclusions $v_{n-1} \in V_{0,(2k+1)(n-2),n-1}$, $v_n \in V_{0,(2k+1)(n-2),n}$ is satisfied but not necessarily both. Furthermore, $(v_{n-2}) \cap S_{(2k+1)(n-2)+2}$ has three (resp. four if $n = 4$) irreducible components: $L_1 v_{n-1}$, $L_1 v_n$ and $(v_{n-2}) \cap V_{0,(2k+1)(n-2)+2,n-3}$ (resp. $(v_{n-2}) \cap V_{0,(2k+1)(n-2)+2,0}$ and $(v_{n-2}) \cap V_{0,(2k+1)(n-2)+2,1}$).

For $n > 4$, we obtain isomorphisms

$$
\begin{aligned}
V_{0,(2k+1)(n-2)+1,n-2} &\rightarrow V_{0,(2k+1)(n-2)+2,n-3} \\
v_{n-2} &\mapsto (v_{n-2}) \cap V_{0,(2k+1)(n-2)+2,n-3}
\end{aligned} \; ,
$$

whereas for $n = 4$ we obtain an isomorphism

$$
\begin{aligned}
V_{0,(2k+1)(n-2)+1,n-2} &\rightarrow V_{0,2k(n-2)+2,0} \times V_{0,(2k+1)(n-2)+2,1} \\
v_{n-2} &\mapsto \left((v_{n-2}) \cap V_{0,(2k+1)(n-2)+2,0}, (v_{n-2}) \cap V_{0,(2k+1)(n-2)+2,1}\right)
\end{aligned} \; .
$$

Moreover, for projective subspaces $U_{n-1} \subseteq P_{(2k+1)(n-2),n-1}, U_n \subseteq P_{(2k+1)(n-2),n}$ the conditions

- $(v_{n-2}) \cap \left(\bigcup_{l>0} V_{l,(2k+1)(n-2)-l+2,n}\right) \subseteq L_1 U_{n-1}$ and $(v_{n-2}) \cap \left(\bigcup_{l>0} V_{l,(2k+1)(n-2)-l+2,n-1}\right) = \emptyset$ are satisfied if and only if $v_{n-2} \in (U_{n-1})$;

- $(v_{n-2}) \cap \left(\bigcup_{l>0} V_{l,(2k+1)(n-2)-l+2,n}\right) = \emptyset$ and $(v_{n-2}) \cap \left(\bigcup_{l>0} V_{l,(2k+1)(n-2)-l+2,n-1}\right) \subseteq L_1 U_n$ are satisfied if and only if $v_{n-2} \in (U_n)$;

- $(v_{n-2}) \cap \left(\bigcup_{l>0} V_{l,(2k+1)(n-2)-l+2,n}\right) \subseteq L_1U_{n-1}$ and $(v_{n-2}) \cap \left(\bigcup_{l>0} V_{l,(2k+1)(n-2)-l+2,n-1}\right) \subseteq L_1U_n$ are satisfied if and only if $v_{n-2} \in (U_{n-1}, U_n)$ but $v_2 \notin (U_{n-1}) \cup (U_n)$.

(6) *Assume that* $3 \leq j \leq n-3$ *(resp.* $j = 2$*) and set* $v_j \in V_{0,(2k+2)(n-2)-j+1,j}$. *Then* $(v_j) \cap \left(\bigcup_{l>1} V_{l,(2k+2)(n-2)-l-j+2,j+1}\right) = \emptyset$. *Furthermore,* $(v_j) \cap S_{(2k+2)(n-2)-j+2}$ *has two (resp. three) irreducible components: a point* $(v_j) \cap V_{1,(2k+2)(n-2)-j+1,j+1}$ *of the form* L_1v_{j+1} *for some* $v_{j+1} \in V_{0,(2k+2)(n-2)-j,j+1}$, *which is the unique element with* $v_j \in (v_{j+1})$, *and another point* $(v_j) \cap V_{0,(2k+2)(n-2)-j+2,j-1}$ *(resp. two other points* $(v_j) \cap V_{0,(2k+2)(n-2)-j+2,0}$, $(v_j) \cap V_{0,(2k+2)(n-2)-j+2,1}$*) providing an isomorphism* $V_{0,(2k+2)(n-2)-j+1,j} \to V_{0,2k(n-2)-j+2,j-1}$ *(resp.* $V_{0,(2k+2)(n-2)-j+1,j} \to V_{0,2k(n-2)-j+2,0} \times V_{0,2k(n-2)-j+2,1}$*). As a consequence, for a projective subspace* $U_{j+1} \subseteq P_{(2k+2)(n-2)-j,j+1}$, *the intersection* $(v_j) \cap \left(\bigcup_{l>0} V_{l,(2k+2)(n-2)-l-j+2,j+1}\right)$ *is contained in* L_1U_{j+1} *if and only if* $v_j \in (U_{j+1})$.

PROOF. For ease of notation in the proof, we will assume that n is even. If n is odd, then the argument works in the same way, except that κ should be applied to the indices when appropriate.

(1) Statement (a) follows immediately from the definition of the cells. For the first part of (b), write $v_0 = \sum_{i=0}^{k} \alpha_i L_{n-1}^{2i} L_n^{2(k-i)}$ and $v_1 = \sum_{i=0}^{k-1} \beta_i L_{n-1}^{2i+1} L_n^{2(k-i)-1}$. The assumptions guarantee that $\sum \alpha_i \neq 0$ and $\sum \beta_i \neq 0$. The image $(v_0, v_1) \cap P_{2k(n-2)+1,2}$, as subset of $Gr(2, S_{2k(n-2)+1}[\rho_2])$, is a projective line and is spanned by

$$L_2v_0 = \sum_{i=0}^{k} \alpha_i L_2 L_{n-1}^{2i} L_n^{2(k-i)} \in L_2(L_{n-1}^2 + L_n^2)^k$$

and

$$L_2v_1 = \sum_{i=0}^{k-1} \beta_i L_2 L_{n-1}^{2i+1} L_n^{2(k-i)-1} \in L_2 L_{n-1} L_n (L_{n-1}^2 + L_n^2)^{k-1}.$$

In particular, if $(v_0, v_1) \cap V_{1,2k(n-2)} = \{L_1v_2\}$, then there exist vectors v_x, v_y in the two-dimensional vector space v_2 satisfying $v_y = \tau(v_x)$, as well as $a_x, b_x, a_y, b_y \in \mathbb{C}$, so that

(6.2)
$$L_1v_x = a_x L_{2,x} v_0 + b_x L_{2,x} v_1,$$
$$L_1v_y = a_y L_{2,y} v_0 + b_y L_{2,y} v_1.$$

In v_0 the highest powers of x and y are $x^{2k(n-2)}$ and $y^{2k(n-2)}$, both with coefficient $\sum_i \alpha_i$. In v_1 the highest powers of x and y are $x^{2k(n-2)}$ and $y^{2k(n-2)}$, the first has coefficient $\sum_i \beta_i$, the second has coefficient $-\sum_i \beta_i$. We apply $L_{2,x}$ to these. In order for the sum to avoid the cell $V_{0,2k(n-2)+1}$, the coefficients must satisfy $[a_x : b_x] = [\sum_i \beta_i : -\sum_i \alpha_i]$, since in this case the coefficient of $x^{2k(n-2)+1}$ is 0. Then the linear combination is necessarily in $V_{1,2k(n-2)}$. Similarly, when applying $L_{2,y}$, the required coefficients are $[a_y : b_y] = [\sum_i \beta_i : \sum_i \alpha_i]$, so we have $a_x b_y = -a_y b_x$.

Therefore, $b_y L_{2,y} L_1 v_x - b_x L_{2,x} L_1 v_y = b_y a_x L_{2,y} L_{2,x} v_0 - b_x a_y L_{2,x} L_{2,y} v_0 = L_1 v_0$, where we have used that $L_{2,x} L_{2,y} = L_1$. As a consequence, $b_y L_{2,y} v_x + b_x L_{2,x} v_y = v_0$ and similarly $-a_y L_{2,y} v_x + a_x L_{2,x} v_y = v_1$, i.e. $v_0, v_1 \in (v_2)$. This proves the first part of (b). The second part of (b), concerning projective subspaces, follows immediately from the first part.

(2) For the first part of the statement let $v_2 = \sum_{i=0}^{2k} \varepsilon_i L_2 L_{n-1}^i L_n^{2k-i} = L_2 \left(\sum_{i=0, i \text{ even}}^{2k} \varepsilon_i L_{n-1}^i L_n^{2k-i} + \sum_{i=0, i \text{ odd}}^{2k} \varepsilon_i L_{n-1}^i L_n^{2k-i} \right) = L_2(v_{\text{even}} + v_{\text{odd}})$. If $n \neq 4$, then by applying L_2 again we get $L_2^2(v_{\text{even}} + v_{\text{odd}}) = (L_1 + L_3)(v_{\text{even}} + v_{\text{odd}})$, where the first sum is operator sum, and the second is vector sum. So $L_2 v_2[\rho_0] = \{L_1 v_{\text{even}}\}$ and $L_2 v_2[\rho_1] = \{L_1 v_{\text{odd}}\}$. If $(v_2) \cap \left(\bigcup_{l>1} V_{l,2k(n-2)-l+2,1} \right) = L_1 v_0$ and $(v_2) \cap \left(\bigcup_{l>1} V_{l,2k(n-2)-l+2,0} \right) = \emptyset$, then $v_{\text{even}} = v_0$ and $v_{\text{odd}} = 0$. The second case is just the opposite, and when $(v_2) \cap \left(\bigcup_{l>1} V_{l,2k(n-2)-l+2,1} \right) = L_1 v_0$ and $(v_2) \cap \left(\bigcup_{l>1} V_{l,2k(n-2)-l+2,0} \right) = L_1 v_1$, then $v_{\text{even}} = a v_0$ and $v_{\text{odd}} = b v_1$ for some coefficients $a, b \in \mathbb{C}$. If $n = 4$, then $L_2^2(v_{\text{even}} + v_{\text{odd}}) = (L_1 + L_3 + L_4)(v_{\text{even}} + v_{\text{odd}})$, and the rest is very similar to the first case.

For the second part of the statement observe that the results so far imply that $V_{0,2k(n-2)+1,2}$ stratifies into disjoint, locally closed subspaces

$$V_{0,2k(n-2)+1,2} = \bigsqcup_{(v_{c_1}, v_{c_2}) \in P_{2k(n-2),c_1} \times P_{2k(n-2),c_2}} \left((v_{c_1}, v_{c_2}) \setminus ((v_{c_1}) \cup (v_{c_2})) \cap V_{0,2k(n-2)+1,2} \right)$$

$$\bigsqcup_{v_{c_2} \in V_{0,2k(n-2),c_2}} \left((v_{c_2}) \cap V_{0,2k(n-2)+1,2} \right)$$

$$\bigsqcup_{v_{c_1} \in V_{0,2k(n-2),c_1}} \left((v_{c_1}) \cap V_{0,2k(n-2)+1,2} \right).$$

The subset $\bigcup_{(v_{c_1}, v_{c_2}) \in V_{0,2k(n-2),c_1} \times V_{0,2k(n-2),c_2}} \left((v_{c_1}, v_{c_2}) \setminus ((v_{c_1}) \cup (v_{c_2})) \cap V_{0,2k(n-2)+1,2} \right)$ is dense in the third stratum, since $V_{0,2k(n-2),c_1} \times V_{0,2k(n-2),c_2}$ is dense in $P_{2k(n-2),c_1} \times P_{2k(n-2),c_2}$. The first statement of (2) implies that the second statement is valid if v_2 is in this subset of $V_{0,2k(n-2)+1,2}$. Similarly, the first statement implies the second statement on the loci $\bigsqcup_{v_{c_1} \in V_{0,2k(n-2),c_1}} \left((v_{c_1}) \cap V_{0,2k(n-2)+1,2} \right)$ and $\bigsqcup_{v_{c_2} \in V_{0,2k(n-2),c_2}} \left((v_{c_2}) \cap V_{0,2k(n-2)+1,2} \right)$. For v_2 in the closed complement of the union of these loci, the second statement follows from the linearity (and thus continuity) of the solution of (6.2), since the U_i are projective.

(3) The statements in this case follow similarly to (2) by observing that $L_2 L_j = L_1 L_{j-1} + L_{j+1}$ (resp. $L_2 L_{n-2} = L_1 L_{n-3} + L_{n-1} + L_n$).

The cases (4), (5) and (6) are analogous. $\qquad \square$

Consider a full block in position $(0, m)$ with $m = k(n-2) + 1$, with label j which is 2 or $n - 2$. In positions $(0, k(n-2))$ and $(1, m)$, above and to the left of this full block, are divided blocks with labels (c_1, c_2), either $(0, 1)$ or $(n-1, n)$. The next lemma gives $P_{m,j}$ the structure of a join of two projective subspaces.

LEMMA 6.27.

(1) *The morphisms $V_{0,k(n-2),c_i} \to V_{0,m,j}$ constructed in Proposition 6.26 extend to morphisms*

$$\phi_i \;:\; P_{k(n-2),c_i} \;\to\; P_{m,j}$$
$$v \;\mapsto\; (v) \cap P_{m,j}.$$

The morphism ϕ_i is injective with image $N_{c_i}^0 := \mathrm{im}(\phi_i) \subset P_{m,j}$ such that $N_{c_1}^0$, $N_{c_2}^0$ are disjoint projective linear subspaces of $P_{m,j}$.

(2) *The join of the disjoint linear subspaces $N_{c_1}^0, N_{c_1}^0 \subset P_{m,j}$ is $P_{m,j}$ itself. Thus given $(v_1, v_2) \in P_{k(n-2),c_1} \times P_{k(n-2),c_2}$, there is a projective line $\mathbb{P}^1 \cong v_1 v_2 \subset P_{m,j}$ containing both $\phi_i(v_i)$, namely, the line defined by v_1, v_2 with endpoints $\phi_i(v_i)$. The lines $v_1 v_2$ cover $P_{m,j}$. For all $(v_1, v_2), (v_1', v_2') \in P_{k(n-2),c_1} \times P_{k(n-2),c_2}$, the intersection $v_1 v_2 \cap v_1' v_2'$ can only be at a common endpoint.*

(3) *For all $(v_1, v_2) \in P_{k(n-2),c_1} \times P_{k(n-2),c_2}$, the intersection $v_1 v_2 \cap V_{0,m,j}$ is*
   - *either empty, exactly when $v_1 \notin V_{0,k(n-2),c_1}$ and $v_2 \notin V_{0,k(n-2),c_2}$;*
   - *or an affine line otherwise.*

PROOF. (1) is immediate. (2) then follows from $\dim P_{k(n-2),c_1} + \dim P_{k(n-2),c_i} + 1 = \dim P_{m,j}$ and Lemma 6.20. (3) is again immediate. $\qquad\square$

As we did already in the statement above, we will sometimes omit the inclusion maps ϕ_i; thus, for subspaces $U_1 \subseteq P_{k(n-2),c_1}$ and $U_2 \subseteq P_{k(n-2),c_2}$, we will denote by $J(U_1, U_2) \subseteq P_{m,j}$ the join of $\phi_1(U_1)$ and $\phi_1(U_1)$ in $P_{m,j}$.

Let

$$M_{c_i}^0 := \phi_i(V_{0,k(n-2),c_i}) \subset V_{0,m,j};$$

these are disjoint affine linear subspaces of the affine space $V_{0,m,j}$. Also consider

$$N_{c_i} = J(P_{k(n-2),c_i}, \overline{P}_{k(n-2),c_{3-i}}) \subset P_{m,j}.$$

This is the locus of points in $P_{m,j}$ covered by lines $v_1 v_2$ one of whose endpoints is at a point "at infinity", in $\overline{P}_{k(n-2),c_{3-i}} = P_{k(n-2),c_{3-i}} \setminus V_{0,k(n-2),c_{3-i}}$. Let

$$M_{c_i} = N_{c_i} \cap V_{0,m,j} \subset V_{0,m,j}$$

be the intersection with the large affine cell of $P_{m,j}$.

LEMMA 6.28. *There exists morphisms $\psi_i \colon M_{c_i} \to M_{c_i}^0$, given by associating to a point*

$$v \in M_{c_i} \subset V_{0,m,j}$$

the "non-infinity" endpoint of the (mostly unique) line $v_1 v_2$ passing through it. The maps ψ_i are trivial vector bundles over affine spaces.

PROOF. See Lemma 6.21. $\qquad\square$

COROLLARY 6.29.

(1) For $i = 1, 2$ the decomposition $P_{k(n-2),c_{3-i}} = \bigsqcup_{(k',l')\in B_{k(n-2),c_{3-i}}} V_{k',l',c_{3-i}}$ induces a decomposition into locally closed subspaces

$$(6.3) \quad M_{c_i} \setminus M_{c_i}^0 = \bigsqcup_{(k',l')\in B_{k(n-2),c_{3-i}}\setminus(1,m)} \left((J(V_{0,k(n-2),c_i}, V_{k',l',c_{3-i}}) \cap V_{0,m,j}) \setminus M_{c_i}^0 \right).$$

(2) Taking into account the bijections $B_{k(n-2),c_i} \cong B_{k(n-2)+2,c_{3-i}}$ and the decomposition (6.3), the space $V_{0,m,j}$ decomposes into locally closed subspaces as

$$V_{0,m,j} = V_{0,m,j}(1, m, 1, m)$$

$$\bigsqcup \left(\bigsqcup_{(k_1,l_1)\in B'_{k(n-2)+2,c_1}} V_{0,m,j}(k_1, l_1, 1, m) \right)$$

$$\bigsqcup \left(\bigsqcup_{(k_2,l_2)\in B'_{k(n-2)+2,c_2}} V_{0,m,j}(1, m, k_2, l_2) \right),$$

where we introduced the notations
- $B'_{k(n-2)+2,c_i} = (B_{k(n-2)+2,c_i} \cup \{\emptyset\}) \setminus \{(1,m)\}$;
- $V_{0,m,j}(\emptyset, 1, m) = M_{c_1}^0$;
- $V_{0,m,j}(k_1, l_1, 1, m) = (J(V_{0,k(n-2),c_1}, V_{k_1,l_1,c_2}) \cap V_{0,m,j}) \setminus M_{c_1}^0$;
- $V_{0,m,j}(1, m, \emptyset) = M_{c_2}^0$;
- $V_{0,m,j}(1, m, k_2, l_2) = (J(V_{0,k(n-2),c_2}, V_{k_2,l_2,c_1}) \cap V_{0,m,j}) \setminus M_{c_2}^0$;
- $V_{0,m,j}(1, m, 1, m) = V_{0,m,j} \setminus (M_{c_1} \cup M_{c_2})$.

The meaning of the notations is that $V_{0,m,j}(k_1, l_1, k_2, l_2)$ consists of exactly those points $v \in V_{0,m,j}$ such that $(v) \cap P_{k(n-2)+2,c_i} \in V_{k_i,l_i,c_i}$ for $i = 1, 2$. The symbol $\emptyset$ at an argument replacing a pair (k_i, l_i) means that there is no such intersection.

6.6. Proofs of propositions about incidence vareties

Here we prove the propositions announced in 6.2. The arguments for Propositions 6.10, 6.11 and 6.13 are very similar, so we will spell out the proof for one of these. The discussion will also prepare the ground for the proof of Proposition 6.12, which is substantially more complicated.

Consider first the situation of 6.13. Namely, $m \not\equiv 0, 1 \bmod n - 2$ is a positive integer, $V_{0,m,j}$ the cell of a full block, c is the label of the full block immediately above the position $(0, m)$, $S_c \subseteq B_{m+1,c}$ is a nonempty maximal subset, and $S \subseteq B_{m,j}$ is a maximal subset which is *allowed* by S_c.

PROOF OF PROPOSITION 6.13. By Proposition 6.26 (3) and (6), for an arbitrary $U \in V_S$, $(U, U_c) \in X_S^{S_c}$ if and only if $U \subseteq (L_1^{-1}U_c) \cap V_{0,m,j}$. Moreover, the composition of L_1^{-1} and the isomorphism $V_{0,m-1,c} \to V_{0,m,j}$ gives an isomorphism $V_{1,m,c} \to V_{0,m,j}$. Hence, for a pair $(\overline{U}, U_c) \in Y_{\overline{S}}^{S_c}$, we have

$$\{U \in V_S|_{\overline{U}} : (U, U_c) \in X_S^{S_c}\} = \left((L_1^{-1}U_c) \cap V_{0,m,j} \right)/\overline{U},$$

and there is a canonical quotient map $V_{0,m,j}/\overline{U} \to V_{1,m,c}/U_c$.

Let us define two families parameterized by $V_{\overline{S}} \times V_{S_c,c}$. The family $\mathcal{F}_c$ is defined to have the fiber $(L_1^{-1}U_c) \cap V_{0,m,j}$ over a pair $(\overline{U}, U_c) \in V_{\overline{S}} \times V_{S_c,c}$. This is a family of affine subspaces of $V_{0,m,j}$. The family $\overline{\mathcal{F}}$ is defined to have the fiber $\overline{U} \subset P_{m,j}$ over a pair $(\overline{U}, U_c) \in V_{\overline{S}} \times V_{S_c,c}$. This is a family of projective subspaces contained in $\overline{P}_{m,j}$. Since the tautological bundle over any Schubert cell in any Grassmannian is trivial, the two families are trivial with affine, respectively projective space fibres. Consider these families over the subset $Y_{\overline{S}}^{S_c} \subset V_{\overline{S}} \times V_{S_c,c}$. By construction, over each point of $Y_{\overline{S}}^{S_c}$, the fibre of $\overline{\mathcal{F}}$ is a subspace of the projective closure of the fiber of $\mathcal{F}_c$ over the same point. In particular, we can take quotients fiberwise. By the considerations above,

$$X_S^{S_1,S_2} = \mathcal{F}_c/\overline{\mathcal{F}}.$$

Moreover, the morphism $\omega \times \mathrm{Id}\colon X_S^{S_1,S_2} \to Y_{\overline{S}}^{S_c}$ over a pair $(\overline{U}, U_c)$ is given by the quotient morphism $V_{0,m,j}/\overline{U} \to V_{1,m,c}/U_c$ times the identity. This shows (1) and (2).

The injectivity statement (3) follows again from the isomorphism $V_{1,m,c} \cong V_{0,m,j}$ given by L_1, since for every pair $(U, U_c) \in X_S^{S_1,S_2}$ one has $U_c = (U, \overline{U}_c) \cap V_{1,m,c}$. □

Consider now the situation of Proposition 6.12; thus $m \equiv 1 \bmod n-2$ is a positive integer, c_1 and c_2 are the labels of the divided block immediately above the block at position (m,j), $S_1 \subseteq B_{m+1,c_1}$, $S_2 \subseteq B_{m+1,c_2}$ are nonempty subsets at least one of which is maximal, and $S \subseteq B_{m,j}$ is a maximal subset, which is allowed by S_1 and S_2.

LEMMA 6.30. *For $i = 1,2$ fix $U_i \in V_{S_i,c_i}$.*

(a) *For an arbitrary $U \in V_S$, $(U, U_1, U_2) \in X_S^{S_1,S_2}$ if and only if $U \subseteq J(\phi_1(L_1^{-1}U_1), \phi_2(L_1^{-1}U_2)) \cap V_{0,m,j}$.*

(b) *If $(\overline{U}, U_1, U_2) \in Y_{\overline{S}}^{S_1,S_2}$, then $\overline{U} \subseteq J(\phi_1(L_1^{-1}U_1), \phi_2(L_1^{-1}U_2)) \cap V_{0,m,j}$.*

(c) *If $(\overline{U}, U_1, U_2) \in Y_{\overline{S}}^{S_1,S_2}$, then*

$$\{U \in V_S|_{\overline{U}} : (U, U_1, U_2) \in X_S^{S_1,S_2}\} = \left(J(\phi_1(L_1^{-1}U_1), \phi_2(L_1^{-1}U_2)) \cap V_{0,m,j}\right)/\overline{U}.$$

PROOF. (a) By Proposition 6.26 for any pair of vectors $(v_1, v_2) \in U_1 \times U_2$ those points of $P_{m,j}$ for which $(v_1, v_2) \cap P_{m+1,c_i}$ is either v_i or empty are exactly those which are on $J(\phi_1(L_1^{-1}v_1), \phi_2(L_1^{-1}v_2))$. Hence, to satisfy the conditions U has to be a subset of

$$\bigcup_{(v_1,v_2)\in U_1\times U_2} J(\phi_1(L_1^{-1}v_1), \phi_2(L_1^{-1}v_2)) \cap V_{0,m,j} = J(\phi_1(L_1^{-1}U_1), \phi_2(L_1^{-1}U_2)) \cap V_{0,m,j}.$$

(b) If $(\overline{U}, U_1, U_2) \in Y$, then $(\overline{U}) \cap P_{m+1,c_i} \subseteq U_i$. Hence, $\phi_i(L_1^{-1}((\overline{U}) \cap P_{m+1,c_i})) \subseteq \phi_i(L_1^{-1}U_i)$, and

$$J(\phi_1(L_1^{-1}((\overline{U}) \cap P_{m+1,c_1})), \phi_2(L_1^{-1}((\overline{U}) \cap P_{m+1,c_2}))) \subseteq J(\phi_1(L_1^{-1}U_1), \phi_2(L_1^{-1}U_2)).$$

By Proposition 6.26 there is an isomorphism $V_{1,m-1,j} \cong V_{0,m-1,c_1} \times V_{0,m-1,c_1}$ in such a way that

$$\overline{U} \cap V_{1,m-1,j} \subseteq J(\phi_1(L_1^{-1}((\overline{U}) \cap P_{m+1,c_1})), \phi_2(L_1^{-1}((\overline{U}) \cap P_{m+1,c_2}))) \cap V_{1,m-1,j}.$$

Similarly, on each cell $V_{k,l,j}$ such that $k + l = m$ and $k \geq 1$, the affine subspace $\overline{U} \cap V_{k,l,j}$ is a subvariety of $J(\phi_1(L_1^{-1}((\overline{U}) \cap P_{m+1,c_1})), \phi_2(L_1^{-1}((\overline{U}) \cap V_{m+1,c_2}))) \cap V_{k,l,j}$. All these mean that $\overline{U} \subseteq J(\phi_1(L_1^{-1}U_1), \phi_2(L_1^{-1}U_2))$.

(c) Recall, that $\overline{U}$ also represents a subspace at infinity for $V_{0,m,j}$, and $V_S|_{\overline{U}} = V_{0,m,j}/\overline{U}$. In fact, we can take the quotient of an arbitrary subspace of $V_{0,m,j}$, whose closure in $P_{m,j}$ contains $\overline{U}$ with respect to (an arbitrary affine subspace representing) $\overline{U}$. Then the statement follows from (a) and (b).   □

PROOF OF PROPOSITION 6.12. It follows from the definitions that $(\omega \times \mathrm{Id} \times \mathrm{Id})(X_S^{S_1,S_2}) \subseteq Y_{\overline{S}}^{S_1,S_2}$. The surjectivity will follow from the calculation of the fibers.

We will define three families of subspaces in $P_{m,j}$ over $V_{\overline{S}} \times V_{S_1,c_1} \times V_{S_2,c_2}$. For $i = 1, 2$ the family $\mathcal{F}_i$ is defined to have the fiber $\phi_i(L_1^{-1}U_i) \subseteq P_{m,j}$ over a three-tuple $(\overline{U}, U_1, U_2) \in V_{\overline{S}} \times V_{S_1,c_1} \times V_{S_2,c_2}$. Let the third family $\mathcal{F}$ has the fiber $\overline{U} \subseteq P_{m,j}$ over the same element. This is of course empty, if $|S| = 1$. It is important to note, that in all cases the fibers are always projective subspaces of $P_{m,j}$.

By Lemma 6.27, there is an embedding $\phi_i \circ L_1^{-1} : P_{m+1,c_{3-i}} \to N_{c_i}^0 \subset P_{m,j}$. Apply this embedding on the fibers of the projectivization of the tautological bundle over the Schubert cell $V_{S_{3-i}}$. Then multiply the base with $V_{\overline{S}} \times V_{S_i}$, and extend the family into this direction as a constant. This gives the bundle $\mathcal{F}_i$. Again, by the fact that the tautological bundle over any Schubert cell is trivial it follows that the $\mathcal{F}_i$'s are also trivial, that is, $\mathcal{F}_i \cong \mathbb{P}^{|S_i|-1} \times V_{\overline{S}} \times V_{S_1} \times V_{S_2}$. Similarly, $\mathcal{F} \cong \mathbb{P}^{|S|-2} \times V_{\overline{S}} \times V_{S_1} \times V_{S_2}$.

By Lemma 6.23, the join of trivial families over a common base is a trivial family of the joins of the fibers:

$$\begin{aligned} J(\mathcal{F}_1, \mathcal{F}_2) &= J(\mathbb{P}^{|S_1|-1} \times V_{\overline{S}} \times V_{S_1} \times V_{S_2}, \mathbb{P}^{|S_2|-1} \times V_{\overline{S}} \times V_{S_1} \times V_{S_2}) \\ &\cong J(\mathbb{P}^{|S_1|-1}, \mathbb{P}^{|S_2|-1}) \times V_{\overline{S}} \times V_{S_1} \times V_{S_2} \\ &\cong \mathbb{P}^{|S_1|+|S_2|-1} \times V_{\overline{S}} \times V_{S_1} \times V_{S_2} \subseteq P_{m,j} \times V_{\overline{S}} \times V_{S_1} \times V_{S_2}. \end{aligned}$$

Therefore, $J(\mathcal{F}_1, \mathcal{F}_2) \cap (V_{0,m,j} \times V_{\overline{S}} \times V_{S_1} \times V_{S_2})$ is a trivial family of affine subspaces of $V_{0,m,j}$ over $V_{\overline{S}} \times V_{S_1} \times V_{S_2}$.

By Lemma 6.30 (b) $\mathcal{F}$ is a (trivial) subfamily of $J(\mathcal{F}_1, \mathcal{F}_2)$ over $Y_{\overline{S}}^{S_1,S_2}$. By Lemma 6.30 (c) $X_S^{S_1,S_2}$ can be constructed as

$$X_S^{S_1,S_2} = (J(\mathcal{F}_1, \mathcal{F}_2) \cap (V_{0,m,j} \times V_{\overline{S}} \times V_{S_1} \times V_{S_2}))/\mathcal{F}|_{Y_{\overline{S}}^{S_1,S_2}}.$$

Hence, $X_S^{S_1,S_2}$ is a trivial family of affine spaces of dimension $|S_1| + |S_2| - |S|$, since it is the quotient of a trivial affine family of fibre dimension $|S_1| + |S_2| - 1$ by another trivial affine family of fibre dimension $|S| - 1$.   □

CHAPTER 7

Type D_n: special loci

In this chapter we analyze those special points of the equivariant Hilbert scheme where a salient block of the associated Young wall fails to contain a generator of the corresponding ideal. More precisely, the case we are interested in is when the intersection of an ideal with a salient cell of its Young wall is already generated by the subspaces of the ideal sitting in cells in lower rows. For this we will describe the special points on the equivariant Grassmannian and then on the incidence varieties.

7.1. Support blocks

We need to analyze the cases when a cell corresponding to a salient block (see Def. 6.14) of a Young wall Y fails to contain a generator of a corresponding ideal $I \in Z_Y$. As an example, recall once again Example 6.5, where the divided missing blocks at position $(1, 3)$ are salient blocks of Y_3, but the corresponding cells do not necessarily contain generators of an ideal $I \in Z_{Y_3}$, i.e. they are already generated by the "parts" of I which are in the cells of the blocks in the lower rows. That this phenomenon can happen at all is one of the main sources of difficulty in our analysis of the strata of the singular Hilbert scheme. We will introduce the notion of a support block for a salient block below. Intuitively, the intersection of an ideal I with the cell corresponding to the support block can generate the affine subspace given by the intersection of I with the cell corresponding to the salient block (such as the support block at position $(0, 3)$ for Y_3), and thus the salient block contains no *new* generator of I. We will make this statement more precise in the rest of this chapter.

We start with some combinatorial preliminaries. Recall the setup of Proposition 6.12: $m \equiv 1 \bmod n - 2$; c_1 and c_2 are the labels of the divided block immediately above the block of label j at position $(m, 0)$; $S_1 \subseteq B_{m+1,c_1}$ and $S_2 \subseteq B_{m+1,c_2}$ are nonempty subsets at least one of which is maximal; $S \subseteq B_{m,j}$ is a maximal subset which is allowed by S_1 and S_2.

For a half-block b of S_i, consider the following two conditions.

(1) The blocks below or to the left of b are not contained in S.
(2) The block below b is contained in S, the complementary half-block b' is contained in S_{3-i}, and the block to the left of their position in not contained in S.

For $i, j = 1, 2$, let us denote by $S_i^{s,j} \subset S_i$ the subset of half-blocks of label c_i satisfying condition (j). Let moreover $S_i^s = S_i^{s,1} \cup S_i^{s,2}$.

The next lemma, whose proof is immediate, connects the global Definition 6.14 for a Young wall Y with the local conditions (1)-(2) where we consider only the m-th and $m+1$-st diagonals for a particular m, and index sets S, S_1 and S_2 as above.

LEMMA 7.1. *Given a Young wall* $Y \in \mathcal{Z}_\Delta$, *let* S, *respectively* S_1 *and* S_2 *denote the set of missing blocks, respectively half-blocks of* Y *on the* m-*th and* $(m+1)$-*st diagonals. The blocks* $S_i^s \subset S_i$ *are exactly the salient blocks of* Y *of label* c_i *on the* $(m+1)$-*st diagonal.*

Thus we can legitimately call the blocks in S_i^s salient blocks in this local situation. Let us introduce the following subsets of S.

- S^l consists of blocks $b \in S$ that are directly to the left of a divided block with labels (c_1, c_2).
- $S^{b,0}$ consists of blocks $b \in S$, so that b is immediately below a divided block with labels (c_1, c_2), the block immediately up-left of b is not in S, and both of the divided blocks above b are in $S_1 \cup S_2$.
- S^{b,c_i} consists of blocks $b \in S$ that are immediately below a divided block, so that the block immediately up-left of b is not in S, and the block of label c_{3-i} above b is in S_{3-i}.
- $S^{b,c_1 \cup c_2}$ consists of blocks $b \in S$ such that b lies immediately below a divided block, and the block immediately up-left of b is contained in S.
- $S^b = S^{b,0} \cup S^{b,c_1} \cup S^{b,c_2} \cup S^{b,c_1 \cup c_2}$.

Note that by the Young wall rules, we necessarily have $S^b = S \setminus S^l$. We will call the blocks in the set $S^{c_i} = S^{b,0} \cup S^{b,c_i} \cup S^{b,c_1 \cup c_2}$ *support blocks for label* c_i. We will define a support relation from S^{c_i} to S_i^s in the next section.

7.2. Special loci in orbifold strata and the supporting rules

Let $Y \in \mathcal{Z}_\Delta$ be a Young wall with a salient block in its bottom row in position $(0, m)$ with $m \equiv 1 \mod n - 2$, a full block immediately below a divided block with labels (c_1, c_2). As before, let S, respectively S_1 and S_2 denote the set of missing blocks, respectively half-blocks of Y on the m-th and $(m+1)$-st diagonals with the corresponding labels.

We introduce index sets depending on S, S_1 and S_2. We consider two cases.

If $\overline{S}$ is not maximal, then let

$$I(S, S_1, S_2) = \left\{ (k_1, l_1, k_2, l_2) \; : \; \begin{array}{l} (k_i, l_i) \in S_i^s \cup \{\emptyset\} \text{ for } i = 1, 2, \\ \text{and at least one } (k_i, l_i) = (1, m) \end{array} \right\}.$$

We partition this index set into the following (possibly empty) disjoint subsets:

- $I(S, S_1, S_2)_0 = \{(k_1, l_1, k_2, l_2) \in I(S, S_1, S_2) : (k_i, l_i) \notin \{\emptyset, (1, m)\}$ for some $i = 1, 2\}$;
- $I(S, S_1, S_2)_1 = \{(1, m, \emptyset), (\emptyset, 1, m)\} \cap I(S, S_1, S_2)$;
- $I(S, S_1, S_2)_{-1} = \{(1, m, 1, m)\} \cap I(S, S_1, S_2)$.

If $\overline{S}$ is maximal, then let

$$I(S, S_1, S_2) = \{(k_1, l_1, k_2, l_2) : (k_i, l_i) \in S_i^s \cup \{\emptyset\} \text{ for } i = 1, 2\}.$$

We remark that in this case $(1, m) \notin S_i^s$ for both $i = 1, 2$. The index set $I(S, S_1, S_2)$ in this case can be partitioned into the following subsets:

- $I(S, S_1, S_2)_0 = \{(k_1, l_1, k_2, l_2) \in I(S, S_1, S_2) : (k_i, l_i) \neq \emptyset \text{ for some } i = 1, 2\}$;
- $I(S, S_1, S_2)_1 = \{(\emptyset, \emptyset)\}$.

As before, $\emptyset$ is used as a symbol replacing a pair in these defintions.

For projective subspaces $P_1 \subseteq P_2 \subseteq P_{m+1,c}$ we introduce the following notation. $(P_2 \setminus P_1) \dashv V_{k,l,c}$ if and only if $(P_2 \setminus P_1) \cap V_{k,l,c} \neq \emptyset$ and k is maximal with this property. This is the smallest cell whose intersection with P_2 is larger than that with P_1.

Recall the truncated Young wall $\overline{Y}$ and the morphism $T \colon Z_Y \to Z_{\overline{Y}}$ from 6.3. The following statement will be proved below in 7.4.

THEOREM 7.2. *There is a decomposition into locally closed subspaces*

$$Z_Y = \bigsqcup_{(k_1, l_1, k_2, l_2) \in I(S, S_1, S_2)} Z_Y(k_1, l_1, k_2, l_2),$$

where

$$Z_Y(k_1, l_1, k_2, l_2) = \{I \in Z_Y : ((I \cap P_{m,j}) \setminus (I \cap \overline{P}_{m,j})) \cap P_{m+1,c_i} \dashv V_{k_i, l_i, c_i} \text{ for } i = 1, 2\}.$$

The symbol $\emptyset = (k_i, l_i)$ means that there is no intersection with P_{m+1,c_i}. Moreover, if $(k_1, l_1, k_2, l_2) \in I(S, S_1, S_2)_e$, then the nonempty fibers of $T \colon Z_Y(k_1, l_1, k_2, l_2) \to Z_{\overline{Y}}$ have Euler characterestic e.

The space $Z_Y(k_1, l_1, k_2, l_2)$ should be thought as the space of those ideals, where the generator in the cell of support block at position $(0, m)$ has an image on the $(m + 1)$-st diagonal at the cells V_{k_i, l_i, c_i} for $i = 1, 2$ which does not come from the rows above. We remark that the mentioned support block in the bottom row is also salient since it is the first missing block in its row. But if one attaches further rows to the bottom of Y, then it may become non-salient.

Recall from Section 7.1 the set $S^{c_i} = S^{b,0} \cup S^{b,c_i} \cup S^{b,c_1 \cup c_2}$ of support blocks for label c_i. In the light of the definition of the sets $I(S, S_1, S_2)$ and Theorem 7.2 we will say that the blocks in S^{c_i} can *support* the salient blocks of label c_i in S_i^s in the following sense.

Each support block $b \in S$ for label c_i can support at most one salient block of label c_i above and to the left of b on the $(m+1)$-st antidiagonal. More precisely, this supporting relationship has to respect the following *supporting rules*.

- Each block in $S^{b,0}$ supports precisely one or two salient blocks, at most one from each label, and at least one of these has to be immediately above it;
- each block in S^{b,c_i} can support at most one salient block of label c_i which is not immediately above it;
- each block in $S^{b,c_1 \cup c_2}$ can support none, one or two salient blocks, at most one from each label, and neither of these is immediately above it.

In this way we define a correspondence from a subset of S^{c_1} to S_1^s and one from a subset of S^{c_2} to S_2^s but these two have to satisfy the restrictions mentioned above on the intersection $S^{c_1} \cap S^{c_2} = S^{b,0} \cup S^{b,c_1 \cup c_2}$. Neither correspondence has to be surjective or be defined on the whole domain, but, where they are defined, they should be injective.

We shall call a salient block $b' \in S_i$ of label c_i *supported*, if the number of support blocks for label c_i in S which are below b is at least as much as the total number of salient blocks of label c_i in S_i counted from the top left, including b' itself. A supported salient block b' satisfying condition (2) above, so that there is a support block in S immediately below b', will be called *directly supported*. The others will be called *non-directly supported*. The supporting relationship will be globalized for the whole diagram in the notion of closing datum, to be defined in 8.2 below.

Recall that during the inductive process in the proof of Theorem 6.1, at each step a new generator appears in the cell corresponding to the salient block in the bottom row. Assume that for $I \in Z_Y$, $(I \cap P_{m,j}) \cap P_{m+1,c_i} = I \cap P_{m+1,c_i}$ for $i = 1, 2$. In this case we will say that *there is no generator of label c_i on the $(m+1)$-st antidiagonal*. Let S, S_1 and S_2 be the index sets for $V_{0,m,j}$, V_{1,m,c_1} and V_{1,m,c_2} respectively. Then using inductively Theorem 7.2 for each block $b \in S^{c_i}$ we get that there is at most one block $b_i \in S_i^s$ such that when the row of b is added to the Young wall, the new generator in the cell of b has nontrivial image in the cell of b_i. Conversely, for each block $b_i \in S_i^s$ there corresponds a support block $b \in S^{c_i}$ determined by I. In particular, this implies

COROLLARY 7.3. *Assume that for $I \in Z_Y$, $(I \cap P_{m,j}) \cap P_{m+1,c_i} = I \cap P_{m+1,c_i}$ for some $i = 1, 2$. Let S, S_1 and S_2 be the index sets for $V_{0,m,j}$, V_{1,m,c_1} and V_{1,m,c_2} respectively.*

(1) $|S_i^{u,1}| \le |S^{b,c_i}| + |S^{b,c_1 \cup c_2}|$;
(2) every salient block of label c_i is supported;
(3) to each salient block of label c_i there corresponds a unique support block for label c_i in the way described above.

7.3. Special loci in Grassmannians

We prepare the ground for the proof of Theorem 7.2 by analyzing the incidence varieties of 6.2 in the case $m \equiv 1 \bmod n - 2$. Once again, we use the notations of Proposition 6.12. The composition with the projection from $V_S \times V_{S_1,c_1} \times V_{S_2,c_2}$ to its first factor, followed by the affine linear fibration $\omega : V_S \to V_{\overline{S}}$, defines a projection map $p_{V_{\overline{S}}} : V_S \times V_{S_1,c_1} \times V_{S_2,c_2} \to V_{\overline{S}}$.

For $i = 1, 2$ let $S_i(\overline{U}) = \{(k_i, l_i) \in B_{m+1,c_i} : (\overline{U}) \cap V_{k_i,l_i,c_i} = \emptyset\}$ be the blocks in the partial profile of $(\overline{U}) \cap P_{m+1,c_i}$ on the $(m+1)$-st diagonal. Then the index sets $I(S, S_1(\overline{U}), S_2(\overline{U}))$ and $I(S, S_1(\overline{U}), S_2(\overline{U}))_e$ introduced above make sense. The following lemma stratifies the fibers of the affine linear fibration $\omega : V_S \to V_{\overline{S}}$.

LEMMA 7.4. *For any $\overline{U} \in V_{\overline{S}}$, there is a stratification*

$$V_{0,m,j}/\overline{U} = \bigsqcup_{(k_1,l_1,k_2,l_2) \in I(S,S_1(\overline{U}),S_2(\overline{U}))} V_{0,m,j}(k_1,l_1,k_2,l_2)/\overline{U},$$

where

$$V_{0,m,j}(k_1,l_1,k_2,l_2)/\overline{U} = \{U \in V_{0,m,j}/\overline{U} : ((U)\setminus(\overline{U}))\cap P_{m+1,c_i} \dashv V_{k_i,l_i,c_i} \text{ for } i = 1,2\}.$$

Moreover, if $(k_1,l_1,k_2,l_2) \in I(S,S_1(\overline{U}),S_2(\overline{U}))_e$, then the space $V_{0,m,j}(k_1,l_1,k_2,l_2)/\overline{U}$ is of Euler characterestic e.

PROOF. We have to distinguish the cases when $\overline{S}$ is maximal or not. The latter case is significantly simpler, so we start with that.

If $\overline{U} \in V_{\overline{S}}$ with $\overline{S}$ not maximal, then $(M_{c_1} + \overline{U}) \cap (M_{c_2} + \overline{U}) = \emptyset$ since M_{c_1} and M_{c_2} are distinct parallel hyperplanes in $V_{0,m,j}$, and there are affine subspaces U_i representing $\overline{U}$ such that $U_i \subseteq M_{c_i}$. Recall from Corollary 6.29 the stratification of $V_{0,m,j}$ which basically comes from the join structure on its closure $P_{m,j}$. This induces a decomposition of $V_{0,m,j}/\overline{U}$ into non-empty, locally closed, but not necessarily disjoint spaces

$$V_{0,m,j}/\overline{U} = ((V_{0,m,j} \setminus (M_{c_1} \cap M_{c_2}))/\overline{U}) \cup (M_{c_1}/\overline{U}) \cup (M_{c_1}/\overline{U})$$

$$\bigcup \left(\bigcup_{(k_1,l_1) \in B_{m+1,c_1}} ((J(V_{0,m-1,c_1}, V_{k_1,l_1,c_2}) \cap V_{0,m,j}) \setminus M_{c_1}^0 + \overline{U})/\overline{U} \right)$$

$$\bigcup \left(\bigcup_{(k_2,l_2) \in B_{m+1,c_1}} ((J(V_{0,m-1,c_2}, V_{k_2,l_2,c_1}) \cap V_{0,m,j}) \setminus M_{c_2}^0 + \overline{U})/\overline{U} \right).$$

Consider a block $(k_i, l_i) \in B_{m+1,c_i} \setminus S_i(\overline{U})$. Then the intersection $(\overline{U}) \cap V_{k_i,l_i,c_i} \neq \emptyset$. Assume that there is an $U \in V_{0,m,j}/\overline{U}$ such that $((U) \setminus (\overline{U})) \cap V_{k_i,l_i,c_i} \neq \emptyset$. Then $\dim((U) \cap V_{k_i,l_i,c_i}) > \dim((\overline{U}) \cap V_{k_i,l_i,c_i})$ so by Lemma 5.6 there is at least one other block in a row above k_i which has a trivial intersection with $(\overline{U})$ but a nontrivial

one with (U). Hence, for any $(k_i, l_i) \in B_{m+1,c_i} \setminus S_i(\overline{U})$ we have

$$\{U \in V_{0,m,j}/\overline{U} \; : \; ((U) \setminus (\overline{U})) \cap P_{m+1,c_i} \dashv V_{k_i,l_i,c_i}\} = \emptyset.$$

On the other hand, if $(k_i, l_i) \in S_i(\overline{U}) \cup \{\emptyset\}$ then

$$((J(V_{0,m-1,c_i}, V_{k_i,l_i,c_{3-i}}) \cap V_{0,m,j}) \setminus M_{c_i}^0 + \overline{U})/\overline{U} =$$
$$\{U \in V_{0,m,j}/\overline{U} \; : \; ((U) \setminus (\overline{U})) \cap P_{m+1,c_i} \dashv V_{k_i,l_i,c_i} \text{ and } ((U) \setminus (\overline{U})) \cap P_{m+1,c_{3-i}} \dashv V_{1,m,c_{3-i}}\}.$$

By dimension constrains these spaces are disjoint and it is easy to see that together with $(V_{0,m,j} \setminus (M_{c_1} \cup M_{c_2}))/\overline{U}, M_{c_1}^0/\overline{U}$, and $M_{c_2}^0/\overline{U}$ they cover $V_{0,m,j}/\overline{U}$. Thus we get a stratification

$$V_{0,m,j}/\overline{U} = V_{0,m,j}(1, m, 1, m)/\overline{U}$$

$$\bigsqcup \left(\bigsqcup_{(k_1,l_1) \in (S_1(\overline{U}) \cup \{\emptyset\}) \setminus \{(1,m)\}} V_{0,m,j}(k_1, l_1, 1, m)/\overline{U} \right)$$

$$\bigsqcup \left(\bigsqcup_{(k_2,l_2) \in (S_2(\overline{U}) \cup \{\emptyset\}) \setminus \{(1,m)\}} V_{0,m,j}(1, m, k_2, l_2)/\overline{U} \right).$$

In particular, there is a stratification

$$(7.1) \qquad M_{c_{3-i}}/\overline{U} = \bigsqcup_{(k_i,l_i) \in (S_i(\overline{U}) \cup \{\emptyset\}) \setminus \{(1,m)\}} V_{0,m,j}(k_i, l_i, 1, m)/\overline{U}.$$

Being an affine space, $M_{c_i}^0/\overline{U}$ has Euler characteristic 1 for $i = 1, 2$. By Lemma 6.22 the spaces $(J(V_{0,k(n-2),c_i}, V_{k_i,l_i,c_{3-i}}) \cap V_{0,m,j}) \setminus M_{c_i}^0$ have Euler characteristic 0, and the same is true for $((J(V_{0,k(n-2),c_i}, V_{k_i,l_i,c_{3-i}}) \cap V_{0,m,j}) \setminus M_{c_i}^0 + \overline{U})/\overline{U}$. This last step follows from the fact that the subspace $\overline{U} \subset \overline{P}_{m,j}$ avoids both the image of $V_{k_i,l_i,c_{3-i}}$ and $M_{c_i}^0$.

If $\overline{U} \in V_{\overline{S}}$ such that $\overline{S}$ is maximal, then $(M_{c_1} + \overline{U}) = (M_{c_2} + \overline{U}) = V_{0,m,j}$ since $\overline{U}$ is transversal to M_{c_1} and M_{c_2}. Therefore, there are two stratifications for $V_{0,m,j}/\overline{U}$ with $i = 1, 2$ as in (7.1). The claimed stratification is the largest common refinement of these two. In particular, there are three types of strata. First, if $U \in ((J(V_{0,k(n-2),c_1}, V_{k_1,l_1,c_2}) \cap V_{0,m,j}) \setminus M_{c_1}^0 + \overline{U}) \cap ((J(V_{0,k(n-2),c_2}, V_{k_2,l_2,c_1}) \cap V_{0,m,j}) \setminus M_{c_2}^0 + \overline{U})/\overline{U}$ for arbitrary $(k_1, l_1) \in S_1(\overline{U})$ and $(k_2, l_2) \in S_2(\overline{U})$, then $((U) \setminus (\overline{U})) \cap P_{m+1,c_i} \dashv V_{k_i,l_i,c_i}$ for $i = 1, 2$. Second, if $U \in (((J(V_{0,k(n-2),c_1}, V_{k_i,l_i,c_{3-i}}) \cap V_{0,m,j}) + \overline{U}) \setminus \bigcup_{(k_{3-i},l_{3-i}) \in S_{3-i}(\overline{U})} (J(V_{0,k(n-2),c_2}, V_{k_{3-i},l_{3-i},c_i}) \cap V_{0,m,j}) + \overline{U})/\overline{U}$, then $((U) \setminus (\overline{U})) \cap P_{m+1,c_i} \dashv V_{k_i,l_i,c_i}$ but $((U) \setminus (\overline{U})) \cap P_{m+1,c_{3-i}} = \emptyset$. Third, if $U \in ((M_{c_1}^0 + \overline{U}) \cap (M_{c_2}^0 + \overline{U}))/\overline{U}$, then $((U) \setminus (\overline{U})) \cap P_{m+1,c_i} = \emptyset$ for $i = 1, 2$. To sum it up, we get a stratification into locally closed spaces

$$V_{0,m,j}/\overline{U} = \bigsqcup_{\substack{(k_1,l_1) \in S_1(\overline{U}) \cup \{\emptyset\} \\ (k_2,l_2) \in S_2(\overline{U}) \cup \{\emptyset\}}} V_{0,m,j}(k_1, l_1, k_2, l_2)/\overline{U}.$$

The Euler characteristic of the stratum $V_{0,m,j}(\emptyset, \emptyset)/\overline{U} = ((M_{c_1} + \overline{U}) \cap (M_{c_2} + \overline{U}))/\overline{U}$ is 1. It is left to the reader that the others have Euler characteristic 0. $\qquad\square$

Let $\mathcal{I}_S = \{(S_1(\overline{U}), S_2(\overline{U})) \; : \; \overline{U} \in V_{\overline{S}}\}$. Actually $\mathcal{I}_S$ only depends on $\overline{S}$. For each $(S_1', S_2') \in \mathcal{I}_S$, let

$$V_{\overline{S}}(S_1', S_2') = \{\overline{U} \in V_{\overline{S}} \; : \; (S_1(\overline{U}), S_2(\overline{U})) = (S_1', S_2')\}.$$

COROLLARY 7.5. *For a fixed $(S_1', S_2') \in \mathcal{I}(S)$ and $(k_1, l_1, k_2, l_2) \in I(S, S_1', S_2')$ the spaces $V_{0,m,j}(k_1, l_1, k_2, l_2)/\overline{U}$ are isomorphic for every $\overline{U} \in V_{\overline{S}}(S_1', S_2')$. Moreover, they fit together into a locally closed subvariety $V_S(k_1, l_1, k_2, l_2) \subseteq V_S$ which is a trivial family over $V_{\overline{S}}(S_1', S_2')$. Using induction and the fact that the fiber product of locally closed spaces is locally closed, we get that there is a stratification*

$$V_S = \bigsqcup_{(S_1', S_2') \in \mathcal{I}_S} \left(\bigsqcup_{(k_1, l_1, k_2, l_2) \in I(S, S_1', S_2')} V_S(k_1, l_1, k_2, l_2) \right)$$

into locally closed subvarieties. Furthermore, if $(k_1, l_1, k_2, l_2) \in I(S, S_1', S_2')_e$, then the fiber of $\omega \colon V_S(k_1, l_1, k_2, l_2) \to V_{\overline{S}}$ has Euler characteristic e.

PROOF. The triviality of the family $V_S(k_1, l_1, k_2, l_2) \to V_{\overline{S}}(S_1', S_2')$ follows from Lemma 6.23 and the fact that $V_{0,m,j}(k_1, l_1, k_2, l_2)/\overline{U}$ is constructed using (union, intersection and difference of) joins in $P_{m,j}$. The rest of the statement is obvious. $\qquad\square$

7.4. Proof of Theorem 7.2

As before, we fix S, S_1 and S_2. Recall that the fiber of the morphism $\omega \times \text{Id} \times \text{Id} \colon X_S^{S_1, S_2} \to Y_{\overline{S}}^{S_1, S_2}$ over an element $(\overline{U}, U_1, U_2) \in Y_{\overline{S}}^{S_1, S_2}$ is $J(L_1^{-1} U_1, L_1^{-1} U_2)/\overline{U}$.

For $i = 1, 2$ let $S_i(\overline{U}) = \{(k_i, l_i) \in S_i \; : \; (\overline{U}) \cap V_{k_i, l_i, c_i} = \emptyset\}$ be the blocks in the partial profile of $(\overline{U}) \cap P_{m+1, c_i}$ on the $(m+1)$-st diagonal. Then the index sets $I(S, S_1(\overline{U}), S_2(\overline{U}))$ and $I(S, S_1(\overline{U}), S_2(\overline{U}))_e$ are defined. The following lemma, whose proof is the same as that of Lemma 7.4, stratifies the fibers of the affine linear fibration $\omega \times \text{Id} \times \text{Id} \colon X_S^{S_1, S_2} \to Y_{\overline{S}}^{S_1, S_2}$.

LEMMA 7.6. *For any $U_1 \in V_{S_1, c_1}, U_2 \in V_{S_2, c_2}$ the stratification of Lemma 7.4 restricts to a stratification*

$$J(L_1^{-1} U_1, L_1^{-1} U_2)/\overline{U} = \bigsqcup_{(k_1, l_1, k_2, l_2) \in I(S, S_1(\overline{U}), S_2(\overline{U}))} J(L_1^{-1} U_1, L_1^{-1} U_2)(k_1, l_1, k_2, l_2)/\overline{U},$$

where

$$J(L_1^{-1} U_1, L_1^{-1} U_2)(k_1, l_1, k_2, l_2)/\overline{U} =$$

$$\{U \in J(L_1^{-1} U_1, L_1^{-1} U_2)/\overline{U} \; : \; ((U) \setminus (\overline{U})) \cap P_{m+1, c_i} \dashv V_{k_i, l_i, c_i} \text{ for } i = 1, 2\}.$$

Moreover, if $(k_1, l_1, k_2, l_2) \in I(S, S_1(\overline{U}), S_2(\overline{U}))_e$, then the space $J(L_1^{-1} U_1, L_1^{-1} U_2)(k_1, l_1, k_2, l_2)/\overline{U}$ is of Euler characteristic e.

Let $\mathcal{I}_S^{S_1,S_2} = \{(S_1(\overline{U}), S_2(\overline{U})) \; : \; (\overline{U}, U_1, U_2) \in Y_{\overline{S}}^{S_1,S_2}\}$. Actually $\mathcal{I}_S^{S_1,S_2}$ only depends on $\overline{S}$, S_1 and S_2. For each $(S_1', S_2') \in \mathcal{I}_S^{S_1,S_2}$, let

$$Y_{\overline{S}}^{S_1,S_2}(S_1', S_2') = \{(\overline{U}, U_1, U_2) \in Y_{\overline{S}}^{S_1,S_2} \; : \; (S_1(\overline{U}), S_2(\overline{U})) = (S_1', S_2')\}.$$

COROLLARY 7.7. *For fixed* $(k_1, l_1, k_2, l_2) \in I(S, S_1', S_2')$ *the spaces* $J(L_1^{-1}U_1, L_1^{-1}U_2)(k_1, l_1, k_2, l_2)/\overline{U}$ *are isomorphic for every* $\overline{U} \in Y_{\overline{S}}^{S_1,S_2}(S_1', S_2')$. *More-over, they fit together into a locally closed subvariety* $X_S^{S_1,S_2}(k_1, l_1, k_2, l_2) \subseteq X_S^{S_1,S_2}$. *Using induction and the fact that the fiber product of locally closed spaces is locally closed, we get that there is a stratification*

$$X_S^{S_1,S_2} = \bigsqcup_{(S_1',S_2')\in\mathcal{I}_S^{S_1,S_2}} \left(\bigsqcup_{(k_1,l_1,k_2,l_2)\in I(S,S_1',S_2')} X_S^{S_1,S_2}(k_1, l_1, k_2, l_2) \right)$$

into a locally closed subvarieties. Furthermore, if $(k_1, l_1, k_2, l_2) \in I(S, S_1', S_2')_e$, *then the fiber of* $\omega \times \mathrm{Id} \times \mathrm{Id} \colon X_S^{S_1,S_2}(k_1, l_1, k_2, l_2) \to Y_{\overline{S}}^{S_1,S_2}(S_1', S_2')$ *has Euler characteristic* e.

PROOF OF THEOREM 7.2. With all these preparations the proof itself is very easy. We just observe that $Z_Y(k_1, l_1, k_2, l_2)$ consist of those points in Z_Y, which map in (6.1) to $X_S^{S_1,S_2}(k_1, l_1, k_2, l_2)$ for some $(S_1', S_2') \in \mathcal{I}_S^{S_1,S_2}$ such that $(k_1, l_1, k_2, l_2) \in I(S, S_1', S_2')$. The result then follows from Corollary 7.7. $\square$

CHAPTER 8

Type D_n: decomposition of the coarse Hilbert scheme

In this chapter we describe some distinguished subsets of the set of Young walls $\mathcal{Z}_\Delta$ of type D_n. They will consist of Young walls which are the analogues of the 0-generated partitions from 4.2. Then we stratify the coarse Hilbert scheme. As always, in the type D case there are substantial extra complications. Hence, we first give a short guide to the chapter.

8.1. Guide to Chapter 8

Since the current chapter is rather technical, and it contains several definitions and constructions, we present first a short guide explaining the motivation of the definitions and also the main steps of the results. For the precise statements and proofs see the body of the chapter in Sections 8.2-8.5. The reader is advised to return to this guide during or after reading these later parts as well. The lemmas and theorems are recollected with the same numbering as in the main body of the chapter.

The injective set theoretical map $i_*\colon \mathrm{Hilb}(\mathbb{C}^2/G_\Delta)^T \to \mathrm{Hilb}([\mathbb{C}^2/G_\Delta])^T = \sqcup_{Y\in\mathcal{Z}_\Delta} Z_Y$ induces a stratification

$$\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)^T = \sqcup_{Y\in\mathcal{Z}_\Delta} W_Y,$$

where $W_Y := i_*^{-1}(Z_Y \cap \mathrm{im}(i_*))$. Set $\widetilde{W}_Y = i_*(W_Y)$.

We define the following subsets of $\mathcal{Z}_\Delta$.

(1) $\mathcal{Z}'_\Delta$ contains those Young walls Y such $W_Y \neq \emptyset$, or equivalently, $\widetilde{W}_Y \neq \emptyset$;
(2) $\mathcal{Z}^1_\Delta$ these are called 0-generated Young walls (plays no role in Chapter 8);
(3) $\mathcal{Z}^0_\Delta$ these are called distinguished 0-generated Young walls.

We have $\mathcal{Z}^0_\Delta \subset \mathcal{Z}^1_\Delta \subset \mathcal{Z}'_\Delta$. Neither of these is closed under the operation of bottom row removal, which is the inductive step used in the proof of Theorem 6.3.

The characterization of W_Y or the computation of $\chi(W_Y)$ is harder. It turns out to group them into subsets indexed by $\mathcal{Z}^0_\Delta$.

The main result of the chapter is

THEOREM 8.14.

$$\sum_{m=0}^{\infty} \chi\left(\mathrm{Hilb}^m(\mathbb{C}^2/G_\Delta)\right) q^m = \sum_{Y\in\mathcal{Z}^0_\Delta} q^{\mathrm{wt}_0(Y)}.$$

LEMMA 8.1. *For each $Y \in \mathcal{Z}_\Delta$, there is a unique $Y' \in \mathcal{Z}'_\Delta$ that contains Y and is minimal with this property with respect to containment.*

LEMMA 8.2. *There is a combinatorial reduction map* red: $\mathcal{Z}'_\Delta \to \mathcal{Z}^0_\Delta$, *restricting to the identity on $\mathcal{Z}^0_\Delta \subset \mathcal{Z}'_\Delta$, which associates to each $Y \in \mathcal{Z}'_\Delta$ a distinguished 0-generated Young wall* red$(Y) \in \mathcal{Z}^0_\Delta$ *of the same 0-weight.*

For a Young wall $Y \in \mathcal{Z}^0_\Delta$, let Rel$(Y) = $ red$^{-1}(Y)$ which is called the set of relatives of Y.

In order to remedy the fact the sets $\mathcal{Z}^0_\Delta$ and $\mathcal{Z}'_\Delta$ are not closed under the operation of bottom row removal we extend these families further and introduce the following notions.

(1) A G_Δ-invariant ideal I is *possibly invariant* if it is generated by functions of character $\rho_0, \rho_1, \rho_{n-1}$ and ρ_n "except for the bottom row". The Young walls of these are denoted as $\mathcal{Z}^P_\Delta$.

(2) A G_Δ-invariant ideal I is *almost invariant* if it is in $\mathbb{C}[x,y]^{G_\Delta}$ "except for the bottom row". The Young walls of these are denoted as $\mathcal{Z}^A_\Delta$. In particular, $\mathcal{Z}'_\Delta \subset \mathcal{Z}^A_\Delta \subset \mathcal{Z}^P_\Delta$.

(3) $\mathcal{Z}^{0,A}_\Delta \subset \mathcal{Z}^A_\Delta$ is a special subset of Young walls, which satisfy conditions similar to that of $\mathcal{Z}^0_\Delta$. In particular, $\mathcal{Z}^0_\Delta \subset \mathcal{Z}^{0,A}_\Delta$.

There is analog of Lemma 8.2 for $\mathcal{Z}^P_\Delta$ and $\mathcal{Z}^{0,A}_\Delta$, including a reduction $\mathcal{Z}^P_\Delta \to \mathcal{Z}^{0,A}_\Delta$ and the notion of relatives. Let Rel$(Y) = $ red$^{-1}(Y)$. Furthermore, the main advantage of these new sets is the following.

LEMMA 8.7. *The sets $\mathcal{Z}^P_\Delta$ and $\mathcal{Z}^{0,A}_\Delta$ are closed under the operation of bottom row removal.*

One has the following commutative diagram:

$$
\begin{array}{ccccc}
\mathcal{Z}^0_\Delta & \hookrightarrow & \mathcal{Z}'_\Delta & & \\
\downarrow & & \downarrow & & \\
\mathcal{Z}^{0,A}_\Delta & \hookrightarrow & \mathcal{Z}^A_\Delta & \hookrightarrow & \mathcal{Z}^P_\Delta.
\end{array}
$$

The relatives of a Young wall $Y \in \mathcal{Z}^0_\Delta$ are the same in $\mathcal{Z}'_\Delta$ and in $\mathcal{Z}^A_\Delta$, but there may be some new relatives in $\mathcal{Z}^P_\Delta$ which are not in $\mathcal{Z}^A_\Delta$. This will cause no problem.

For each $Y \in \mathcal{Z}^P_\Delta$, there is a locally closed decomposition:

$$
Z_Y = \bigsqcup_{d \in pcd(Y)} Z_Y(d)
$$

Here $pcd(Y)$ is the set of all *partial closing data* defined on the support blocks of Y. These are obtained with the inductive usage of Theorem 7.2 for each possibly invariant ideal.

Accordingly, for each $Y \in \mathcal{Z}_\Delta^A$ the almost invariant ideals are in a locally closed subset $\widetilde{W}_Y \subset Z_Y$ and there is a locally closed decomposition

$$\widetilde{W}_Y = \sqcup_{d \in \mathrm{cd}(Y)} \widetilde{W}_Y(d).$$

The set $\mathrm{cd}(Y)$ of closing data are special partial closing data for Young walls in $\mathcal{Z}_\Delta^A$, in which all salient blocks of label 1, $n-1$ or n are closed, except possibly one on the bottom row.

The main ingredient for the proof of Theorem 8.14 is

PROPOSITION 8.12. *For all $Y \in \mathcal{Z}_\Delta^{0,A}$,*

$$\sum_{Y' \in \mathrm{Rel}(Y)} \chi(\widetilde{W}_{Y'}) = 1.$$

This implies that $\sum_{Y' \in \mathrm{Rel}(Y)} \chi(\widetilde{W}_{Y'}) = 1$ for $Y \in \mathcal{Z}_\Delta^0$, and can be rewritten as

$$\sum_{Y' \in \mathrm{Rel}(Y)} \sum_{d' \in \mathrm{cd}(Y')} \chi(\widetilde{W}_{Y'}(d')) = 1.$$

The statement is proved in several steps.

Fix a Young wall $Y \in \mathcal{Z}_\Delta^P$, and a partial closing datum $d \in \mathrm{pcd}(Y)$. We say that a support block for label c is *of type $e \in \{-1, 0, 1\}$* if, when its row is considered as the bottom row, the associated half blocks according to d are in the set $I(S_1, S_2, S)_e$ in the notation of Theorem 7.2. The total number of support blocks of type e is denoted as $s_e(d)$.

LEMMA 8.9. *If the salient block b at the bottom row of Y is of type $e \in \{-1, 0, 1\}$, then*

$$\chi(Z_Y(d)) = e \cdot \chi(Z_{\overline{Y}}(\overline{d})).$$

PROPOSITION 8.10. *If $Y \in \mathcal{Z}_\Delta^P$ and $d \in \mathrm{pcd}(Y)$, then (with the notation $0^0 = 1$)*

$$\chi(Z_Y(d)) = (-1)^{s_{-1}(d)} \cdot 0^{s_0(d)} \cdot 1^{s_1(d)}.$$

COROLLARY 8.11.   *(1) Let $Y \in \mathcal{Z}_\Delta^P$ and $d \in \mathrm{pcd}(Y)$. If Y contains a salient block of any label to which a support block not immediately below it is associated under d, then $\chi(Z_Y(d)) = 0$.*
  (2) Let $Y \in \mathcal{Z}_\Delta^A$ and $d \in \mathrm{cd}(Y)$.
    (a) If Y contains a nondirectly supported salient block of label 1, $n-1$ or n, then $\chi(\widetilde{W}_Y(d)) = 0$.
    (b) If Y does not contain any nondirectly supported salient block of label 1, $n-1$ or n, but d is not contributing, then $\chi(\widetilde{W}_Y(d)) = 0$.

By Corollary 8.11 (2.a), the strata associated to those Young walls $Y' \in \mathrm{Rel}(Y)$ that have at least one undirectly supported salient block of label $n-1$ or n above the bottom row do not contribute to the sum. Therefore we can restrict our attention to the subset of Young walls in which the salient blocks not in the bottom row are

- directly supported salient block-pairs of label $0/1$ or $n-1/n$, or
- arbitrary salient blocks of label 0.

In particular, we can assume that all Young walls we are working with satisfy (R1'): the salient blocks of label $n-1$ or n not in the bottom row are all directly supported. Moreover, by Corollary 8.11 (2.b) we can assume that each closing datum is contributing. That is, for each salient block of label 1, $n-1$ or n not in the bottom row, the associated support block is immediately below it, and this holds also for those label 0 salient blocks which have an associated support block.

Then, we perform a case-by-case consideration.

8.2. Distinguished 0-generated Young walls

For a Young wall $Y \in \mathcal{Z}_\Delta$, denote by $\mathrm{wt}_0(Y)$ the 0-weight of Y, the number of half-blocks labelled 0 in Y.

Recall from 6.3, respectively 7.1 the notions of a salient block and a support block for a given label $c \in \{0, 1, n-1, n\}$; we will use also all other notation introduced in the latter section. We call a pair of salient half-blocks (b, b') sharing the same position a *salient block-pair*.

Consider the following conditions for a Young wall $Y \in \mathcal{Z}_\Delta$.

(A1) All salient blocks of Y are labelled 0, 1, $n-1$ or n.
(A2) Every salient block of Y labelled $c \in \{1, n-1, n\}$ is supported.

Let $\mathcal{Z}'_\Delta \subset \mathcal{Z}_\Delta$ be the set of Young walls Y which satisfy conditions (A1)-(A2). We will prove in Theorem 8.4, that $\mathcal{Z}'_\Delta$ is the set of Young walls for which $Z_Y \cap \mathrm{im}(i_*) \neq \emptyset$.

LEMMA 8.1. *For each $Y \in \mathcal{Z}_\Delta$, there is a unique $Y' \in \mathcal{Z}'_\Delta$ that contains Y and is minimal with this property with respect to containment.*

We will prove this statement at the end of the section.

Given a Young wall $Y \in \mathcal{Z}_\Delta$, a *closing datum* for Y is a function d from the set of the salient blocks of Y of label $c \in \{1, n-1, n\}$, and some *subset* of the salient blocks of Y with label 0, to the set of support blocks of Y, such that

- for each salient block b of label c for which d is defined, the associated support block $d(b)$ is a support block for label c, and lies on the previous antidiagonal and in a lower row than that of b;
- for each fixed $c \in \{0, 1, n-1, n\}$ the different salient blocks of label c are mapped to different support blocks;
- each support block for label c can support at most one salient block of label c according to the supporting rules spelled out at the end of 7.1.

By condition (A2), for every $Y \in \mathcal{Z}'_\Delta$ with a nonempty set of salient blocks the set $\mathrm{cd}(Y)$ of closing data for Y is nonempty. If all salient blocks of Y of label 1, $n-1$ or n are directly supported, then a closing datum $d \in \mathrm{cd}(Y)$ is called *contributing*, if

to every salient block of label on which d is defined, it associates the support block immediately below it.

We define two subsets of $\mathcal{Z}'_\Delta$. Consider the following conditions for a Young wall $Y \in \mathcal{Z}_\Delta$.

(R1) The salient blocks of Y of label $n-1$ or n are all part of a directly supported salient block-pair.

(R2) Y has no salient block with label in the set $\{1, \ldots, n-2\}$.

(R3) Any consecutive series of rows of Y having equal length and ending in half $n-1/n$-blocks is longer than $n-2$, or $n-1$ if the length of the rows is $n-1$, and the last one starts with a block labelled 1 (see Example 8.15 below for the latter condition being broken).

Young walls satisfying (R1)–(R2) will be called 0-*generated*. Let $\mathcal{Z}^1_\Delta \subset \mathcal{Z}_\Delta$ denote the set of 0-generated Young walls. They automatically satisfy (A1)–(A2), so indeed $\mathcal{Z}^1_\Delta \subset \mathcal{Z}'_\Delta$. Let further $\mathcal{Z}^0_\Delta \subset \mathcal{Z}^1_\Delta$ be the set of those Young walls which in addition satisfy (R3) also. These will be called *distinguished*.

LEMMA 8.2. *There is a combinatorial reduction map* $\mathrm{red} \colon \mathcal{Z}'_\Delta \to \mathcal{Z}^0_\Delta$, *restricting to the identity on* $\mathcal{Z}^0_\Delta \subset \mathcal{Z}'_\Delta$, *which associates to each* $Y \in \mathcal{Z}'_\Delta$ *a distinguished* 0-*generated Young wall* $\mathrm{red}(Y) \in \mathcal{Z}^0_\Delta$ *of the same* 0-*weight.*

PROOF. Starting with a Young wall $Y \in \mathcal{Z}'_\Delta$, we construct $\mathrm{red}(Y)$ by enforcing (R1)–(R2) and (R3) in turn, making sure in the second step that (R1)–(R2) remain fulfilled.

First, if a Young wall $Y \in \mathcal{Z}'_\Delta$ violates (R1) or (R2) at a salient block of label 1, $n-1$ or n, find the lowest row where this happens, and extend Y by adding as many extra blocks as possible to this row without modifying the 0-weight. Thus, the extension stops either just before any full block below which there is a missing full block, or just before the next 0 block in the row, whichever comes earlier. Then in the row above this, one or two blocks may become salient. If at least one of these new salient blocks is not of label 0, then we repeat the same procedure. Following this procedure all the way to the top of Y gives a new Young wall which satisfies (R1) and (R2). These moves do not increase the number of places where the Young wall violates (R3).

Second, assume a Young wall Y satisfies (R1) and (R2) but violates (R3): there is a consecutive series of rows having equal length and ending in half $n-1/n$-blocks, but the length of this series is $m \leq n-2$. Remove the half block from end of the lowest row of such a series. Then a new supported salient block-pair appears. If the block b immediately above this block-pair is contained in Y, then we remove b, as well as the blocks to the right of it in order to obtain a valid Young wall. Any full block above b also cannot be present in a valid Young wall, so we remove that also, as well as all blocks to the right. Continue the removal process until there is

a full block above the last removed block. This process terminates after m steps, when it arrives at a row which is already short enough. In this way we decreased the number places where the Young wall violates (R3), but haven't increased the number of places where it violates (R1) or (R2). The 0-weight of the Young wall remains unchanged, since the length of the series was at most $n-2$.

Combining these steps, we obtain a Young wall red(Y) that satisfies (R3) as well as (R1) and (R2), and so lies in $\mathcal{Z}_\Delta^0$, and has the same 0-weight. $\qquad\square$

EXAMPLE 8.3. Let $n = 4$ and let us consider the following six Young walls.
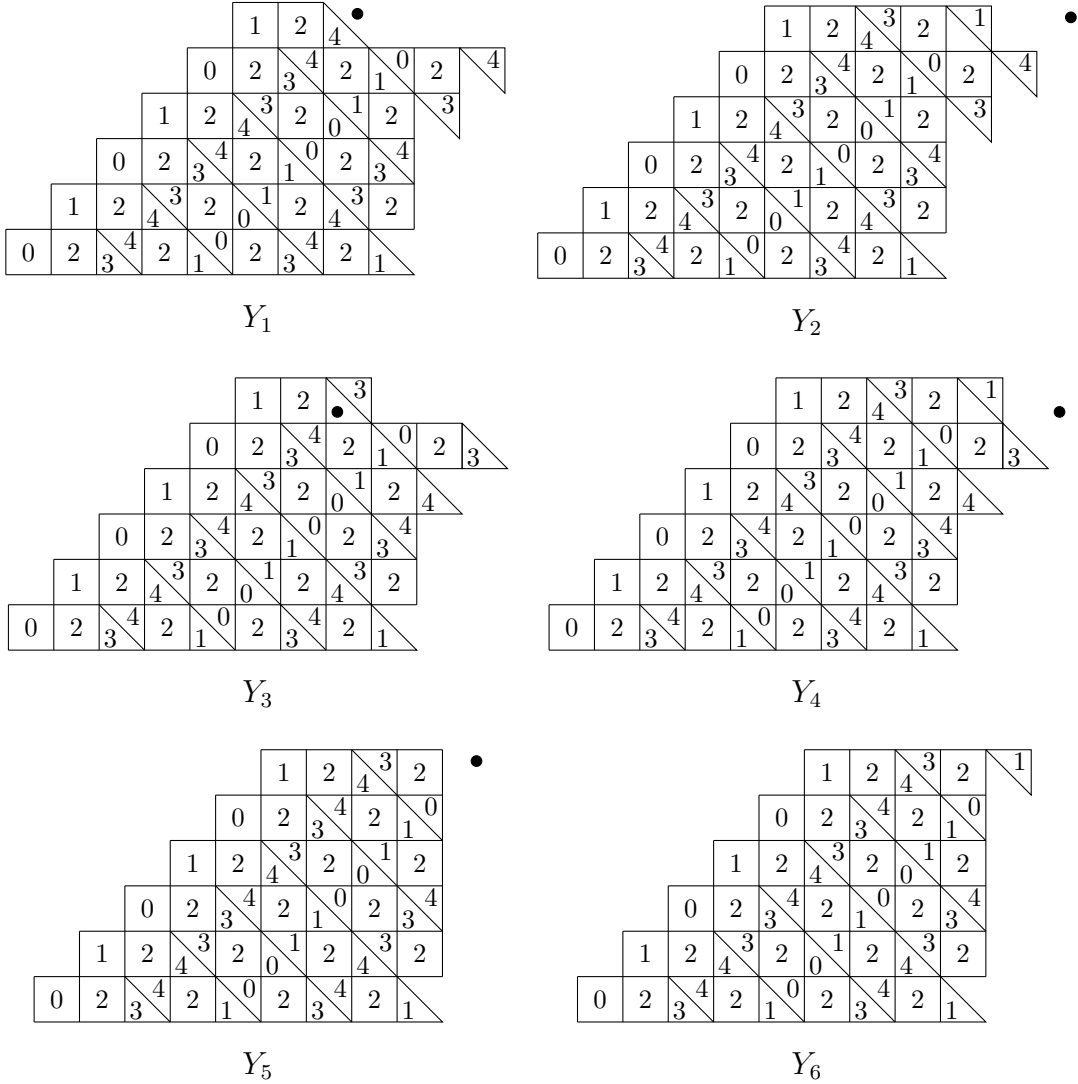
It can be checked that all these satisfy conditions (A1) and (A2), and hence lie in $\mathcal{Z}_\Delta'$. On the other hand, the Young walls Y_1 and Y_3 violate (R1), and are extended to Y_2, resp. Y_4 in the first step of the reduction process. Both of these still violate (R3); in the second step, they become red$(Y_1) = $ red$(Y_3) = Y_6 \in \mathcal{Z}_\Delta^0$. Y_5 satisfies (R1), but violates (R2) and is extended to Y_6 in the first step with red$(Y_5) = Y_6 \in \mathcal{Z}_\Delta^0$ also. In each case the bullets indicate where exactly these violations occur. The blocks without numbers are not contained in the Young walls.

For a Young wall $Y \in \mathcal{Z}_\Delta^0$, let $\mathrm{Rel}(Y) = \mathrm{red}^{-1}(Y)$ denote the set of relatives of Y, the Young walls we can get from Y by the inverses of the reduction steps above. This is a finite directed set, directed by the steps of the proof of Lemma 8.2. In Example 8.3, all Y_i are relatives of Y_6.

PROOF OF LEMMA 8.1. Fix a Young wall $Y \in \mathcal{Z}_\Delta$. The positions of its 0 blocks determine uniquely a Young wall $Y_1 \in \mathcal{Z}_\Delta^0$ such that $\mathrm{wt}_0(Y) = \mathrm{wt}_0(Y_1)$, and the 0 blocks in Y_1 are exactly the 0 blocks in Y. Y_1 does not necessarily contain Y. Consider the set $\mathrm{Rel}_Y(Y_1) \subset \mathcal{Z}_\Delta'$ of those relatives of Y_1 which contain Y. This set is nonempty, since we can always extend Y_1 with the inverse of the move (R3) in Lemma 8.2 until there are only label 0 salient blocks. There can be several of these since there is an ambiguity in the inverse of the move (R3), but there is no Young wall having the same 0 weight as Y_1 which is not contained in at least one of these extended Young walls.

Suppose that $\mathrm{Rel}_Y(Y_1)$ has two distinct minimal elements Y_2, Y_3 with respect to containment. Then there is at least one row ending in a half block, where one of Y_2, Y_3 has a left triangle, and the other has a right triangle, but otherwise the row has the same length. Then the length of the series of successive rows with the same length is the same in the two Young walls. If this length is more than $n - 2$, then they cannot both contain Y. If it is $n - 2$ or less, then Y_2, Y_3 are the two results of the inverse of the move (R3) applied on a smaller Young wall. Since Y was a Young wall, also this smaller Young wall contains Y. Hence, neither of Y_2, Y_3 could be minimal. The same reasoning applies to all places where there is the left triangle/right triangle ambiguity. Thus there is a unique minimal element in the set of relatives of Y_1 containing Y. $\qquad\square$

8.3. The decomposition of the coarse Hilbert scheme

Let us turn to the Hilbert scheme of points on the quotient $\mathbb{C}^2/G_\Delta$, the coarse Hilbert scheme $\mathrm{Hilb}(\mathbb{C}^2/G_\Delta) = \sqcup_n \mathrm{Hilb}^n(\mathbb{C}^2/G_\Delta)$. Recall that the inclusion $\mathbb{C}[x, y]^{G_\Delta} \subset \mathbb{C}[x, y]$ defines a morphism

$$p_* : \mathrm{Hilb}([\mathbb{C}^2/G_\Delta]) \to \mathrm{Hilb}(\mathbb{C}^2/G_\Delta), \quad J \mapsto J^{G_\Delta} = J \cap \mathbb{C}[x, y]^{G_\Delta}$$

and a map of sets

$$i^* : \mathrm{Hilb}(\mathbb{C}^2/G_\Delta)(\mathbb{C}) \to \mathrm{Hilb}([\mathbb{C}^2/G_\Delta])(\mathbb{C}), \quad I \mapsto \mathbb{C}[x, y].I$$

between the coarse and the orbifold Hilbert schemes.

The purpose of this section is to prove the following result.

THEOREM 8.4. *The decomposition of the equivariant Hilbert scheme* $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])$ *from Theorem 6.1 induces a locally closed decomposition*

$$\mathrm{Hilb}(\mathbb{C}^2/G_\Delta) = \bigsqcup_{Y \in \mathcal{Z}_\Delta'} \mathrm{Hilb}(\mathbb{C}^2/G_\Delta)_Y$$

of the coarse Hilbert scheme $\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)$ *into strata indexed bijectively by the set* $\mathcal{Z}'_\Delta$ *of Young walls of type D_n satisfying (A1)-(A2) above. The stratum* $\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)_Y$ *is contained in the m-th Hilbert scheme* $\mathrm{Hilb}^m(\mathbb{C}^2/G_\Delta)$ *for $m = \mathrm{wt}_0(Y)$.*

PROOF. We start with the universal ideal $\mathcal{J} \lhd \mathcal{O}_{\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)} \otimes \mathbb{C}[x,y]^{G_\Delta}$, which exists since $\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)$ is a fine moduli space. Using the relative pullback, we obtain an invariant ideal $\mathbb{C}[x,y].\mathcal{J} \lhd \mathcal{O}_{\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)} \otimes \mathbb{C}[x,y]$, which however is not a *flat* family of invariant ideals over $\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)$. Take the flattening stratification of the base with index set F, to obtain a decomposition

$$(8.1) \qquad \mathrm{Hilb}(\mathbb{C}^2/G_\Delta) = \sqcup_{f \in F} \mathrm{Hilb}(\mathbb{C}^2/G_\Delta)_f$$

over which the restrictions $(\mathbb{C}[x,y].\mathcal{J})_f$ are flat. These flat families of invariant ideals of $\mathbb{C}[x,y]$ define classifying maps

$$i_f \colon \mathrm{Hilb}(\mathbb{C}^2/G_\Delta)_f \to \mathrm{Hilb}^\rho([\mathbb{C}^2/G_\Delta])$$

from these strata to components of the equivariant Hilbert scheme. The latter smooth varieties are decomposed into locally closed strata by Theorem 6.1 as

$$(8.2) \qquad \mathrm{Hilb}([\mathbb{C}^2/G_\Delta]) = \sqcup_{Y \in \mathcal{Z}_\Delta} \mathrm{Hilb}([\mathbb{C}^2/G_\Delta])_Y.$$

The stratification (8.2) gives a stratification on $\mathrm{im}(i_f)$ for each $f \in F$ since over each $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])_Y$ the classifying map is flat. Hence, we can pull it back to obtain a decomposition

$$\mathrm{Hilb}(\mathbb{C}^2/G_\Delta) = \bigsqcup_{Y \in \mathcal{Z}_\Delta} \mathrm{Hilb}(\mathbb{C}^2/G_\Delta)_Y,$$

where we have, set-theoretically,

$$\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)_Y(\mathbb{C}) = \{I \in \mathrm{Hilb}(\mathbb{C}^2/G_\Delta)(\mathbb{C}) : i^*(I) \in \mathrm{Hilb}([\mathbb{C}^2/G_\Delta])_Y(\mathbb{C})\}.$$

The whole construction is compatible with the $T = \mathbb{C}^*$-action, so we can also decompose the T-fixed locus representing homogeneous ideals as

$$\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)^T = \bigsqcup_{Y \in \mathcal{Z}_\Delta} W_Y,$$

where

$$W_Y(\mathbb{C}) = \{I \in \mathrm{Hilb}(\mathbb{C}^2/G_\Delta)(\mathbb{C}) : i^*(I) \in Z_Y(\mathbb{C})\}.$$

Notice also that by construction, the maps i_ρ above are given by the pullback map i^*. In other words, when restricted to the strata $\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)_Y \supset W_Y$, the map i^* becomes a morphism of schemes. On the other hand, it is also clear that, letting $\widetilde{W}_Y$ denote the image of i^* inside Z_Y, p_* and i^* are two-sided inverses and so $W_Y \cong \widetilde{W}_Y \subset Z_Y$.

To conclude, we need to show that for a fixed $I \in \mathrm{Hilb}(\mathbb{C}^2/G_\Delta)$, the Young wall Y associated with the pullback ideal $J = i^*(I)$ necessarily lies in $\mathcal{Z}'_\Delta$. It is clearly enough to assume that I, and so J, are homogeneous. The ideal J, being a pullback, is of course generated by invariant polynomials. On the other hand, as

we have seen during the proof of Theorem 6.1, a homogeneous ideal is generated by polynomials lying in salient blocks. While not all salient blocks necessarily contain a new generator, it is clear that salient blocks labelled j with $2 \leq j \leq n - 2$ must contain a generator. Since such a generator is not allowed in an invariant ideal, Y must satisfy condition (A1).

To discuss the other condition (A2), let us return to the inductive proof of Theorem 6.1. Corollary 7.3 (2) implies that if there is no generator on a given antidiagonal of Y, then the salient blocks of label $c \in \{1, n, n-1\}$ on this antidiagonal are supported. For an invariant ideal, this condition is required to be satisfied for all salient blocks of label $c \in \{1, n, n - 1\}$. This concludes the proof. $\square$

8.4. Possibly and almost invariant ideals

We wish to study the Euler characteristics of the strata of the coarse Hilbert scheme obtained in Theorem 8.4, using the inductive approach used in 6.3 in our study of the orbifold Hilbert scheme. However, as things stand, the setup does not lend itself well to induction based on the removal of the bottom row from a Young wall, since the set of Young walls $\mathcal{Z}'_\Delta$ is clearly not closed under the removal of the bottom row. To remedy this, we introduce two auxiliary constructions. From now on, except when noted, every ideal is supposed to be T-invariant.

First, call an ideal $I \lhd \mathbb{C}[x, y]$ *possibly invariant*, if it is generated by

- polynomials which transform under G_Δ according to ρ_0, ρ_1, ρ_{n-1} or ρ_n,
- and at most one τ-invariant pair of polynomials of the same degree, forming a two-dimensional representation of G_Δ, and not lying in the image of the operator L_1.

Equivalently, the second condition says that the corresponding two-dimensional subspace lies in the large stratum of the appropriate projective space parameterizing such subspaces.

Second, a possibly invariant ideal I will be called *almost invariant*, if it is generated by

- G_Δ-invariant elements,
- and at most a single polynomial, or pair of polynomials of the same degree, forming a one-or two-dimensional representation of G_Δ, and not lying in the image of the operator L_1.

Let us denote by $\mathcal{Z}^P_\Delta \subset \mathcal{Z}_\Delta$ the subset of all Young walls which are characterized by the following condition:

(A1') all salient blocks of Y are labelled 0, 1, $n - 1$ or n, except possibly for a single salient block in the bottom row of a different label.

Moreover, let $\mathcal{Z}^A_\Delta \subset \mathcal{Z}^P_\Delta$ be the set of Young walls which satisfy condition (A1') as well as the following second condition:

(A2') every salient block of Y labelled $c \in \{1, n-1, n\}$ is supported, except possibly the ones in the bottom row.

The following statement follows immediately from our setup.

PROPOSITION 8.5.

(1) *Possibly invariant ideals correspond to points in the strata* $Z_Y \subset \mathrm{Hilb}([\mathbb{C}^2/G_\Delta])$ *where* $Y \in \mathcal{Z}_\Delta^P$.

(2) *Points parameterizing almost invariant ideals lie in constructible subsets* $\widetilde{W}_Y \subset Z_Y$ *of strata* $Z_Y \subset \mathrm{Hilb}([\mathbb{C}^2/G_\Delta])$ *for* $Y \in \mathcal{Z}_\Delta^A$.

By definition, we have $\mathcal{Z}_\Delta' \subset \mathcal{Z}_\Delta^A$. For $Y \in \mathcal{Z}_\Delta'$, the constructible subset provided by Proposition 8.5(2) is exactly $\widetilde{W}_Y = i_*(W_Y)$. Therefore, in the sequel we will denote these strata as $\widetilde{W}_Y$ for arbitrary $Y \in \mathcal{Z}_\Delta^A$ as well. Further, if for $Y \in \mathcal{Z}_\Delta^A$, also $\overline{Y} \in \mathcal{Z}_\Delta^A$, where $\overline{Y}$ is the Young wall obtained by removing from Y the bottom row as in 6.3, then the map $T: Z_Y \to Z_{\overline{Y}}$ of Lemma 6.16 takes $\widetilde{W}_Y$ to $\widetilde{W}_{\overline{Y}}$.

Furthermore, let $\mathcal{Z}_\Delta^{0,A} \subset \mathcal{Z}_\Delta^A$ be the subset defined by the following conditions:

(R1') the salient blocks of label $n-1$ or n not in the bottom row are all directly supported;

(R2') there is no salient block with label in $\{1, \ldots, n-2\}$ except possibly for the bottom row;

(R3') any consecutive series of rows, except the one starting in the bottom row, having equal length and ending in half $n-1/n$-blocks is longer than $n-2$, or $n-1$ if the length of the rows is $n-1$ and the last one starts with a block labelled 1.

LEMMA 8.6. *There is a combinatorial reduction map* $\mathrm{red}: \mathcal{Z}_\Delta^P \to \mathcal{Z}_\Delta^{0,A}$ *associating to each* $Y' \in \mathcal{Z}_\Delta^P$ *a unique* $Y \in \mathcal{Z}_\Delta^{0,A}$.

PROOF. The proof of Lemma 8.2 above goes through unchanged in this setting and the reduction process gives a well-defined element in $\mathcal{Z}_\Delta^P$. After the reduction there is no indirectly supported salient block. Hence, each salient block is directly supported, and in particular supported as required by condition (A2'). Therefore, the output of the reduction is an element in $\mathcal{Z}_\Delta^A$ with the properties (R1')-(R3'). $\square$

Once again, for $Y \in \mathcal{Z}_\Delta^{0,A}$ the Young walls $\mathrm{Rel}(Y) = \mathrm{red}^{-1}(Y)$ will be called the relatives of Y. The following statement is clear from the definitions.

LEMMA 8.7. *The sets* $\mathcal{Z}_\Delta^P$ *and* $\mathcal{Z}_\Delta^{0,A}$ *are closed under the operation of bottom row removal.*

The sets introduced so far can be placed in the following commutative diagram:

$$
\begin{array}{ccccc}
\mathcal{Z}_\Delta^0 & \hookrightarrow & \mathcal{Z}_\Delta' & & \\
\downarrow & & \downarrow & & \\
\mathcal{Z}_\Delta^{0,A} & \hookrightarrow & \mathcal{Z}_\Delta^A & \hookrightarrow & \mathcal{Z}_\Delta^P.
\end{array}
$$

The relatives of a Young wall $Y \in \mathcal{Z}_\Delta^0$ are the same in $\mathcal{Z}_\Delta'$ and in $\mathcal{Z}_\Delta^A$, but there may be some new relatives in $\mathcal{Z}_\Delta^P$. This will cause no problem. Even though we are interested in strata of the Young walls in the upper row, it is easier to work in the lower row because of Lemma 8.7.

The notion of a closing datum generalizes word by word for ideals in the stratum of $Y \in \mathcal{Z}_\Delta^A$. For ideals in the stratum of Young walls in $\mathcal{Z}_\Delta^P$ we have to relax it, since not all salient blocks of label 1, $n-1$ or n are supported. A *partial closing datum* for a Young wall in $\mathcal{Z}_\Delta^P$ is given by associating to some of its salient blocks of label $c \in \{0, 1, n-1, n\}$ a support block for label c in the previous antidiagonal and below the salient block, in such a way that to each support block for label c at most one salient block of label c is associated. We say that those salient blocks to which there is an associated support block are *closed*. The set of all partial closing data for $Y \in \mathcal{Z}_\Delta^P$ will be denoted as $\mathrm{pcd}(Y)$. Closing data are special partial closing data for Young walls in $\mathcal{Z}_\Delta^A$, in which all salient blocks of label 1, $n-1$ or n are closed, except possibly one on the bottom row.

Fix a Young wall $Y \in \mathcal{Z}_\Delta^P$ such that in the bottom row the salient block is a support block for label $c \in \{0, 1, n-1, n\}$. Let $\overline{Y}$ be the truncation of Y. By Lemma 8.7, $\overline{Y} \in \mathcal{Z}_\Delta^P$. If $\overline{d}$ is a partial closing datum associated to some $\overline{I} \in Z_{\overline{Y}}$, then using the decomposition of Theorem 7.2 we can extend it to a partial closing datum for each ideal in the fiber over $\overline{I}$; in particular, if $I \in Z_Y(k_1, l_1, k_2, l_2)$ for some pairs (k_1, l_1) and (k_2, l_2), and either of these is a salient block of label c, then we associate to them the support block in the bottom row. By induction, we obtain

COROLLARY 8.8. *Given $Y \in \mathcal{Z}_\Delta^P$, every $I \in Z_Y$ defines a unique partial closing datum $d(I) \in \mathrm{pcd}(Y)$.*

For $d \in \mathrm{pcd}(Y)$, let $Z_Y(d) \subseteq Z_Y$ be the subset of those ideals which have partial closing datum d. Then

$$
Z_Y = \sqcup_{d \in \mathrm{pcd}(Y)} Z_Y(d)
$$

is a locally closed decomposition of the stratum Z_Y. Similarly, for an element $Y \in \mathcal{Z}_\Delta^A$ let $\widetilde{W}_Y(d) \subseteq \widetilde{W}_Y$ be the subset of those ideals which have closing datum d. Then

$$
\widetilde{W}_Y = \sqcup_{d \in \mathrm{cd}(Y)} \widetilde{W}_Y(d)
$$

is a locally closed decomposition.

8.5. Euler characteristics of strata and the coarse generating series

In this section, we derive information about the topological Euler characteristics of the strata of the coarse Hilbert scheme constructed above.

Fix a Young wall $Y \in \mathcal{Z}_\Delta^P$, and a partial closing datum $d \in \mathrm{pcd}(Y)$. We say that a support block for label c is *of type* $e \in \{-1, 0, 1\}$ if, when its row is considered as the bottom row, the associated half blocks according to d are in the set $I(S_1, S_2, S)_e$ in the notation of Theorem 7.2.

LEMMA 8.9. *Assume that (Y, d) are such such that in the bottom row of Y, the salient block b of label $j \in \{2, n-2\}$ is a support block for label $c \in \{0, 1, n-1, n\}$. Let $(\overline{Y}, \overline{d})$ be the truncation of (Y, d). If the salient block b is of type $e \in \{-1, 0, 1\}$, then*

$$\chi(Z_Y(d)) = e \cdot \chi(Z_{\overline{Y}}(\overline{d})).$$

PROOF. Using the notations of 7.2, let (k_1, l_1, k_2, l_2) be the quadruple of the half blocks associated to the support block, and consider the diagram

$$
\begin{array}{ccccc}
Z_Y(d) & \subseteq & Z_Y & \longrightarrow & \mathcal{X}_Y \\
& & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \omega \times \mathrm{Id}} \\
Z_{\overline{Y}}(\overline{d}) & \subseteq & Z_{\overline{Y}} & \longrightarrow & \mathcal{Y}_{\overline{Y}}.
\end{array}
$$

Returning once again to the process proving Theorem 6.1, when we obtained Z_Y from the truncation $Z_{\overline{Y}}$, we saw that those ideals in Z_Y that does not have a generator in the strata of the missing half blocks at (k_1, l_1) and at (k_2, l_2) are necessarily in $Z_Y(k_1, l_1, k_2, l_2)$ and all ideals in $Z_Y(k_1, l_1, k_2, l_2)$ have this property. Formally, a point of Z_Y over $Z_{\overline{Y}}(\overline{d})$ is in $Z_Y(d)$ if and only if

$$((I \cap P_{m,j}) \setminus (I \cap \overline{P}_{m,j})) \cap P_{m+1,c_i} \dashv V_{k_i, l_i, c_i} \text{ for } i = 1, 2.$$

Under the operation T the space $Z_Y(d)$ necessarily maps onto $Z_{\overline{Y}}(\overline{d})$. Hence, we get that

$$Z_Y(d) = Z_{\overline{Y}}(\overline{d}) \times_{Z_{\overline{Y}}} Z_Y(k_1, l_1, k_2, l_2).$$

By Theorem 7.2 the fibers of T on $Z_Y(k_1, l_1, k_2, l_2)$ have Euler characteristic e. Thus

$$\chi(Z_Y(d)) = e \cdot \chi(Z_{\overline{Y}}(\overline{d})).$$

$\square$

For $e \in \{-1, 0, 1\}$ and $c \in \{0/1, n-1/n\}$ let $s_e(d, c)$ be the number of support blocks for label c of type e, and let $s_e(d) = s_e(d, 0/1) + s_e(d, n-1/n)$. Applying Lemma 8.9 inductively, we get the following.

PROPOSITION 8.10. *For $Y \in \mathcal{Z}_\Delta^P$ and $d \in \mathrm{pcd}(Y)$,*

$$\chi(Z_Y(d)) = (-1)^{s_{-1}(d)} \cdot 0^{s_0(d)} \cdot 1^{s_1(d)},$$

where we adopt the convention $0^0 = 1$.

COROLLARY 8.11.

(1) Let $Y \in \mathcal{Z}_\Delta^P$ and $d \in \mathrm{pcd}(Y)$. If Y contains a salient block of any label to which a support block not immediately below it is associated under d, then $\chi(Z_Y(d)) = 0$.

(2) Let $Y \in \mathcal{Z}_\Delta^A$ and $d \in \mathrm{cd}(Y)$.

    (a) If Y contains a nondirectly supported salient block of label 1, $n-1$ or n, then $\chi(\widetilde{W}_Y(d)) = 0$.

    (b) If Y does not contain any nondirectly supported salient block of label 1, $n-1$ or n, but d is not contributing, then $\chi(\widetilde{W}_Y(d)) = 0$.

PROOF. (1) follows from Proposition 8.10, and both parts of (2) follows from (1). $\qquad\square$

The main ingredient for calculating the coarse generating series is the following statement.

PROPOSITION 8.12. For all $Y \in \mathcal{Z}_\Delta^{0,A}$,

$$\sum_{Y' \in \mathrm{Rel}(Y)} \chi(\widetilde{W}_{Y'}) = 1.$$

Recall from Section 8.4 that the relatives of a Young wall $Y \in \mathcal{Z}_\Delta^0$ are the same in $\mathcal{Z}_\Delta'$ and in $\mathcal{Z}_\Delta^A$, but there may be some new relatives in $\mathcal{Z}_\Delta^P$ which are not in $\mathcal{Z}_\Delta^A$. On the other hand, since $\widetilde{W}_Y = \emptyset$ for $Y \in \mathcal{Z}_\Delta^P \setminus \mathcal{Z}_\Delta^A$ Proposition 8.12 implies

COROLLARY 8.13. For all $Y \in \mathcal{Z}_\Delta^0$,

$$\sum_{Y' \in \mathrm{Rel}(Y)} \chi(\widetilde{W}_{Y'}) = 1.$$

PROOF OF PROPOSITION 8.12. By Corollary 8.11 (2.a), the strata associated to those Young walls $Y' \in \mathrm{Rel}(Y)$ that have at least one undirectly supported salient block of label $n-1$ or n above the bottom row do not contribute to the sum. Therefore we can restrict our attention to the subset of Young walls in which the salient blocks not in the bottom row are

    • directly supported salient block-pairs of label $0/1$ or $n-1/n$, or
    • arbitrary salient blocks of label 0.

In particular, we can assume that all Young walls we are working with satisfy (R1'). Moreover, by Corollary 8.11 (2.b) we can assume that each closing datum is contributing. That is, for each salient block of label 1, $n-1$ or n not in the bottom row, the associated support block is immediately below it, and this holds also for those label 0 salient blocks which have an associated support block.

As in the proof of Theorem 6.3, we will prove the proposition by induction on the number of rows. Fix a Young wall $Y \in \mathcal{Z}_\Delta^{0,A}$. We re-write the statement into the

form

$$\sum_{Y'\in\mathrm{Rel}(Y)} \sum_{d'\in\mathrm{cd}(Y')} \chi(\widetilde{W}_{Y'}(d')) = 1.$$

Let $\overline{Y}$ be the truncation of Y; by Lemma 8.7, $\overline{Y} \in \mathcal{Z}_\Delta^{0,A}$ also.

Assume first that the block above the salient block in the bottom row of Y is not divided. Then the relatives of Y are exactly the extensions of those of $\overline{Y}$ with the bottom row of Y, a closing datum on any such Young wall extends uniquely to the extended Young wall, and the corresponding strata are isomorphic. In this case, the induction step is obvious.

Assume next that the block above the salient block in the bottom row of Y is divided. Let S be the index set of the block at $(0, m)$, S_1 and S_2 the index sets for the blocks at $(1, m)$. The following cases can occur:

(1) $\overline{S}$ is maximal;
(2) $\overline{S}$ is not maximal, and exactly one of S_1 or S_2 is maximal;
(3) $\overline{S}$ is not maximal, and both S_1 and S_2 are maximal.

If $\overline{S}$ is maximal, then to each relative $\overline{Y}' \in \mathrm{Rel}(\overline{Y})$ there corresponds a unique relative $Y' \in \mathrm{Rel}(Y)$ which satisfies (R1') and (R2'): we extend each relative of $\overline{Y}$ with the bottom row of Y. A closing datum $\overline{d}'$ on one of these relatives $\overline{Y}'$ can be extended to a closing datum d' on the corresponding relative of Y by assigning $(\emptyset, \emptyset)$ to the new salient block appearing in the bottom row. By Theorem 7.2, $(\emptyset, \emptyset) \in I(S, S_1, S_2)_1$, and

$$\sum_{Y'\in\mathrm{Rel}(Y)} \sum_{d'\in\mathrm{cd}(Y')} \chi(\widetilde{W}_{Y'}(d')) = \sum_{\overline{Y}'\in\mathrm{Rel}(\overline{Y})} \sum_{\overline{d}'\in\mathrm{cd}(\overline{Y}')} \chi(\widetilde{W}_{\overline{Y}'}(\overline{d}')) = 1.$$

If $\overline{S}$ is not maximal, and S_i is maximal while S_{3-i} is not, then the missing block at $(1, m)$ is not salient, and the subspace in its stratum is necessarily in the image of the subspace at $(0, m)$. Again, to each relative of $\overline{Y}$ there corresponds a unique relative of Y, and

$$\sum_{Y'\in\mathrm{Rel}(Y)} \sum_{d'\in\mathrm{cd}(Y')} \chi(\widetilde{W}_{Y'}(d')) = \sum_{\overline{Y}'\in\mathrm{Rel}(\overline{Y})} \sum_{\overline{d}'\in\mathrm{cd}(\overline{Y}')} \chi(\widetilde{W}_{\overline{Y}'}(\overline{d}')) = 1.$$

Finally, suppose that $\overline{S}$ is not maximal, but S_1 and S_2 are maximal. In this case new relatives appear when we add the bottom row of Y to $\overline{Y}$. We investigate the two possible cases individually.

If the divided block above the new salient block is a full $n-1/n$ block, then for each relative of $\overline{Y}$ there are two other relatives, which contain either the label $(n-1)$ half block or the label n half block. If the relatives of $\overline{Y}$ are $\{\overline{Y}_0 = \overline{Y}, \overline{Y}_1, \overline{Y}_2, \dots\}$, then the relatives of Y are

$$\{Y_0 = Y, Y_1, Y_2, \dots\} \cup \bigcup_{c\in\{n-1,n\}} \{Y_{0,c} = Y_c, Y_{1,c}, Y_{2,c}, \dots\}.$$

We obtain these by performing the inverse of the move (R3) in the algorithm of Lemma 8.2. The addition of the bottom row of Y to an $\overline{Y}_i$ schematically looks like this:
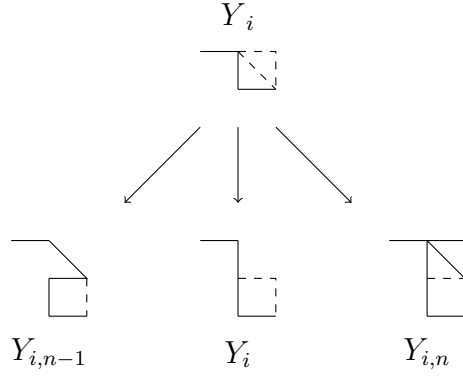
Here the block of label c is in the complement of the Young wall $Y_{i,c}$, while its pair is in it. There is only one contributing closing datum on each of Y_i, $Y_{i,n-1}$ and $Y_{i,n}$ extending a contributing closing datum $\overline{d}_i$ on $\overline{Y}_i$. Namely, to the new support block of Y_i we associate both the divided block above it, and every other part of $\overline{d}_i$ is kept constant. We denote these by d_i, $d_{i,n-1}$ and $d_{i,n}$, respectively.

We claim that for $c \in \{n-1, n\}$,

$$\chi(\widetilde{W}_{Y_{i,c}}(d_{i,c})) = \chi(\widetilde{W}_{\overline{Y}_i}(\overline{d}_i)).$$

To show this, we define a morphism $\widetilde{W}_{\overline{Y}_i}(\overline{d}_i) \to \widetilde{W}_{\overline{Y}_{i,c}}(\overline{d}_{i,c})$, where $\overline{Y}_{i,c}$ is the truncation of $Y_{i,c}$. This morphism is given by the restriction of an ideal $I \in \widetilde{W}_{\overline{Y}_i}(\overline{d}_i)$ to the union of those cells whose block is missing from $\overline{Y}_{i,c}$. The Young wall $\overline{Y}_{i,c}$ has the same salient blocks as $\overline{Y}_i$ except a half block in the bottom row. Hence, the result is necessarily an ideal, which has the same generators as I except for the cell $V_{0,m,c}$ where it does not have any generator. Therefore, the image of this morphism is in $W_{\overline{Y}_{i,c}}(\overline{d}_{i,c})$; here $(0, m)$ is the position of the salient block-pair in the bottom row of $\overline{Y}_i$, which is also the position of the salient half block of label $\kappa(c)$ in the bottom row of $\overline{Y}_{i,c}$.

Assume that there are two ideals $I, I' \in \widetilde{W}_{\overline{Y}_i}(\overline{d}_i)$ which map to the same element of $\widetilde{W}_{\overline{Y}_{i,c}}(\overline{d}_{i,c})$ under this morphism. Then they only differ in the function at $V_{0,m,c}$, or more precisely in the point of $V_{0,m,c}/\overline{U}$ which represents the subspace in $V_{0,m,c}$. Here $\overline{U} = I \cap \overline{P}_{m,c} = I' \cap \overline{P}_{m,c}$. Then any ideal I'' which is their affine linear combination, i.e. whose corresponding subspace in $V_{0,m,c}$ is a linear combination of those of I and I', is also an element of $\widetilde{W}_{\overline{Y}_i}(\overline{d}_i)$, mapped to the same ideal as I and I' under the morphism. In particular, the fibers of the morphism are affine spaces. Taking into account that $\chi(\widetilde{W}_{Y_{i,c}}(d_{i,c})) = \chi(\widetilde{W}_{\overline{Y}_{i,c}}(\overline{d}_{i,c}))$, this proves the claim.

Thus,

$$\chi(\widetilde{W}_{Y_i}(d_i)) + \chi(\widetilde{W}_{Y_{i,n-1}}(d_{i,n-1})) + \chi(\widetilde{W}_{Y_{i,n}}(d_{i,n}))$$
$$= -\chi(\widetilde{W}_{\overline{Y}_i}(\overline{d}_i)) + \chi(\widetilde{W}_{\overline{Y}_i}(\overline{d}_i)) + \chi(\widetilde{W}_{\overline{Y}_i}(\overline{d}_i)) = \chi(\widetilde{W}_{\overline{Y}_i}(\overline{d}_i)),$$

where in the first equality we used Lemma 8.9.

Second, if the divided block above the new salient block is a full $0/1$ block, then for each relative of $\overline{Y}$ there is a new relative which contains the label 1 half block (and possibly some other blocks above it). Let us denote the relatives of $\overline{Y}$ as $\{\overline{Y}_0 = \overline{Y}, \overline{Y}_1, \overline{Y}_2, \dots\}$. Then the relatives of Y are

$$\{Y_0 = Y, Y_1, Y_2, \dots\} \cup \{Y_{0,1}, Y_{1,1}, Y_{2,1}, \dots\}.$$

We obtain these by performing the inverse of the move (R2) in the algorithm of Lemma 8.2. Schematically:

$$\overline{Y}_i$$

$$Y_i \qquad\qquad Y_{i,1}$$
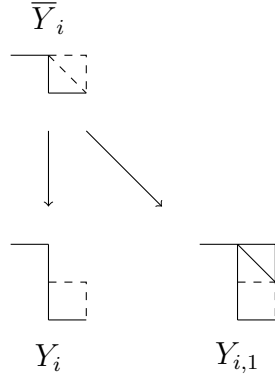
In this case $Y_{i,0}$ cannot appear as a relative, since that would change the 0-weight. Given a contributing closing datum $\overline{d}_i$ on $\overline{Y}_i$, shifting it appropriately gives a unique contributing closing datum $d_{i,1}$ on $Y_{i,1}$. On Y_i, $\overline{d}_i$ induces two contributing closing data:

- d_i is obtained by associating the support block in the bottom row to the salient half-block pair above it;
- d_i' is obtained by associating the support block in the bottom row to the salient half block of label 1 above it only.

Again, using Lemma 8.9 we get

$$\chi(\widetilde{W}_{Y_i}(d_i)) + \chi(\widetilde{W}_{Y_i}(d_i')) + \chi(\widetilde{W}_{Y_{i,1}}(d_{i,1}))$$
$$= -\chi(\widetilde{W}_{\overline{Y}_i}(\overline{d}_i)) + \chi(\widetilde{W}_{\overline{Y}_i}(\overline{d}_i)) + \chi(\widetilde{W}_{\overline{Y}_i}(\overline{d}_i)) = \chi(\widetilde{W}_{\overline{Y}_i}(\overline{d}_i)).$$

Summing over these, we obtain in all cases

$$\sum_{Y' \in \mathrm{Rel}(Y)} \sum_{d' \in \mathrm{cd}(Y')} \chi(\widetilde{W}_{Y'}(d)) = \sum_{\overline{Y}' \in \mathrm{Rel}(\overline{Y})} \sum_{\overline{d}' \in \mathrm{cd}(\overline{Y}')} \chi(\widetilde{W}_{\overline{Y}'}(\overline{d})),$$

which proves the induction step. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Putting everything together, we obtain the following result, the analogue of Corollary 4.3 in type D.

THEOREM 8.14. *Let Δ be of type D_n. Then the generating function of Euler characteristics of the coarse Hilbert schemes of points of the corresponding singular surface $\mathbb{C}^2/G_\Delta$ is given by the combinatorial generating series*

$$\sum_{m=0}^{\infty} \chi\left(\mathrm{Hilb}^m(\mathbb{C}^2/G_\Delta)\right) q^m = \sum_{Y \in \mathcal{Z}_\Delta^0} q^{\mathrm{wt}_0(Y)}.$$

PROOF. We use the decomposition of Theorem 8.4. For $Y \in \mathcal{Z}_\Delta'$, we have $\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)_Y^T = W_Y$ and thus $\chi(W_Y) = \chi(\mathrm{Hilb}(\mathbb{C}^2/G_\Delta)_Y)$. Now use Corollary 8.13 to sum the Euler characteristics of the strata W_Y along the fibres of the combinatorial reduction map red of Lemma 8.2. □

EXAMPLE 8.15. Let $n = 4$ and let $Y \in \mathcal{Z}_\Delta^0$ be the distinguished 0-generated Young wall
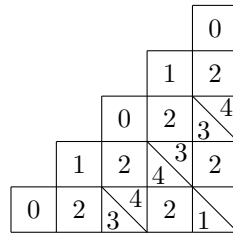
The parameter space Z_Y of this Young wall Y is isomorphic to that of Example 6.6, which in turn is isomorphic to that of Example 6.5. In particular, $Z_Y \cong \mathbb{A}^1$. The difference is that in this case the salient blocks are the 0-labelled blocks at $(0, 4)$ and $(1, 5)$.

Denote by Y_3 and Y_4 the 0-generated non-distinguished Young walls
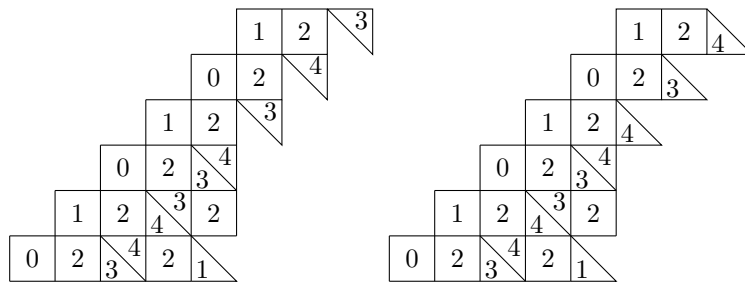
We have $Y_3, Y_4 \in \mathcal{Z}_\Delta'$ and are in fact 0-generated, but both violate condition (R3) at their fourth row, so they are not distinguished. In the first step of the reduction algorithm of Lemma 8.2, we remove the half block of label 3 (resp., 4) from Y_3 (resp, Y_4). Then we remove the blocks of label 2 and 4 (resp., 2 and 3) from the fifth row. Finally we remove the blocks of 1,2 and 3 (reps., 1,2 and 4) from the sixth row. This shows that both Young walls Y_3, Y_4 reduce to Y. It is not difficult to see that these are in fact all the relatives of Y. As explained in Example 6.6, when the two generators

$(v_0, v_0') \in V_{0,4,0} \times V_{1,5,0}$ are in a special position such that $L_1 v_0, v_0' \in (L_1 L_3)$, then the ideal (v_0, v_0') has Young wall Y_3. Similarly, if $L_1 v_0, v_0' \in (L_1 L_4)$, then (v_0, v_0') has Young wall Y_4. In fact we can think of the corresponding strata as gluing to one stratum inside the invariant Hilbert scheme $\mathrm{Hilb}^3(\mathbb{C}^2/D_4)$, the strata of Y_3 and Y_4 "patching in" the gaps of the stratum of Y. At least on the level of Euler characteristics, this is what Proposition 8.12 shows in full generality.

Type D_n: abacus combinatorics

In this chapter we carry out all the combinatorial calculations that are left from the proofs of Theorem 3.2 and Theorem 3.5 in the type D case. Fortunately, an abacus representation of the Young walls in type D has already been developed by other authors [39, 41]. We carry out the enumerations in both the equivariant and the coarse case on the abacus. Along the way, we confine the abacus representation of 0-generated Young walls.

9.1. Guide to Chapter 9

Since the statements and methods in this chapter are again quite technical, we start with a short summary or extended abstract. Again, the lemmas and theorems are recollected with the same numbering as in the main body of the chapter.

Associated with each Young wall $Y \in \mathcal{Z}_\Delta$ there is an abacus configuration on an abacus specific to the type D case. As in the type A case, there are certain sets of blocks called *bars* on Y corresponding to the regular representation of G_Δ. These bars can be removed from Y, and the removals correspond to operations on the associated abacus. A Young wall $Y \in \mathcal{Z}_\Delta$ will be called a core Young wall, if no bar can be removed from it without violating the Young wall rules. Let $\mathcal{C}_\Delta \subset \mathcal{Z}_\Delta$ denote the set of all core Young walls.

PROPOSITION 9.2. *Given a Young wall $Y \in \mathcal{Z}_\Delta$, any complete sequence of bar removals through Young walls results in the same core $\mathrm{core}(Y) \in \mathcal{C}_\Delta$, defining a map of sets*

$$\mathrm{core}\colon \mathcal{Z}_\Delta \to \mathcal{C}_\Delta.$$

The process can be described on the abacus, and results in a bijection

$$\mathcal{Z}_\Delta \longleftrightarrow \mathcal{P}^{n+1} \times \mathcal{C}_\Delta,$$

where $\mathcal{P}$ is the set of ordinary partitions. Finally, there is also a bijection

$$(9.1) \qquad\qquad \mathcal{C}_\Delta \longleftrightarrow \mathbb{Z}^n.$$

THEOREM 9.3. *Let $q = q_0 q_1 q_2^2 \ldots q_{n-2}^2 q_{n-1} q_n$, corresponding to a single bar.*

(1) The multi-weight $\underline{q}^{\underline{wt}(Y)} = \prod_{j=0}^n q_j^{wt_j(Y)}$ of a core Young wall $Y \in \mathcal{C}_\Delta$ corresponding to an element $\overline{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ under the correspondence (9.1) can be expressed as

$$q_1^{m_1} \cdot \cdots \cdot q_n^{m_n} (q^{1/2})^{\overline{m}^\top \cdot C \cdot \overline{m}},$$

where C is the Cartan matrix of type D_n.

(2) The multi-weight generating series

$$Z_\Delta(q_0, \ldots, q_n) = \sum_{Y \in \mathcal{Z}_\Delta} \underline{q}^{\underline{wt}(Y)}$$

of Young walls for Δ of type D_n can be written as

$$Z_\Delta(q_0, \ldots, q_n) = \frac{\displaystyle\sum_{\overline{m}=(m_1,\ldots,m_n)\in\mathbb{Z}^n}^{\infty} q_1^{m_1} \cdots \cdot q_n^{m_n} (q^{1/2})^{\overline{m}^\top \cdot C \cdot \overline{m}}}{\displaystyle\prod_{m=1}^{\infty} (1-q^m)^{n+1}}.$$

The main result of the chapter is

THEOREM 9.8. *Let Δ be of type D_n, and let ξ be a primitive $(2n-1)$-st root of unity. Then the generating series of the set $\mathcal{Z}_\Delta^0$ of distinguished 0-generated Young walls is given in terms of the generating function of all Young walls by the following substitution:*

$$\sum_{Y \in \mathcal{Z}_\Delta^0} q^{\mathrm{wt}_0(Y)} = Z_\Delta(q_0, \ldots, q_n)\Big|_{q_0=\xi^2 q,\, q_1=\cdots=q_n=\xi}.$$

For the proof of this, we first introduce a map

$$p \colon \mathcal{Z}_\Delta \to \mathcal{Z}_\Delta^0,$$

and describe it combinatorially on the abacus.

Besides the abacus representation we introduce another representation of distinguished 0-generated Young walls in terms of sequences of integer pairs. Given $Y \in \mathcal{Z}_\Delta^0$, let t_i denote the total number of beads in the i-th row of its abacus representation, and l_i the number of beads in the rightmost position of the i-th row. We obtain a sequence of pairs $(t_i, l_i)_{i\in\mathbb{Z}_+}$, only finitely many of which do not equal $(0,0)$.

COROLLARY 9.7. *Given $Y \in \mathcal{Z}_\Delta^0$, the associated sequence of pairs $(t_i, l_i)_{i\in\mathbb{Z}_+}$ satisfies the following conditions.*

(1) For all i, $0 \le l_i \le t_i$.

(2) For all i, if $t_i > 0$, then either $l_i = t_i$ is even, or l_i is odd.

(3) Either $\sum_i t_i$ is even, or $t_1 - l_1 = 2n - 4$.

Conversely, any sequence $(t_i, l_i)_{i\in\mathbb{Z}_+}$ satisfying these conditions arises as a sequence associated to at least one Young wall $Y \in \mathcal{Z}_\Delta^0$. More precisely, the number of different Young walls $Y \in \mathcal{Z}_\Delta^0$ corresponding to any given sequence is 2^m, where m is the number of indices i such that $t_i - l_i > 2n - 2$. All Young walls Y corresponding to a single sequence have the same multi-weight, when the weights for labels $n-1$ and n are counted together.

COROLLARY 9.11. *Let* $Y \in \mathcal{Z}_\Delta^0$ *be a distinguished 0-generated Young wall described by the data* $\{(t_i, l_i)_i\}$. *Then*

$$\sum_{Y' \in p_*^{-1}(Y)} q^{\underline{\mathrm{wt}}(Y')} \Big|_{q_1 = \cdots = q_n = \xi, \, q_0 = \xi^2 q} = q^{\mathrm{wt}_0(Y)} \xi^{\sum_{j \neq 0} (\mathrm{wt}_j(Y) - \dim \rho_j \cdot \mathrm{wt}_0(Y)) - \sum_i c(t_i, l_i)},$$

where $c(t_i, l_i)$ *is an integer depending only on* t_i *and* l_i.

Then we show by reducing every $Y \in \mathcal{Z}_\Delta^0$ to a distinguished 0-generated core that

$$\xi^{\sum_{j \neq 0} (\mathrm{wt}_j(Y) - \dim \rho_j \cdot \mathrm{wt}_0(Y)) - \sum_i c(t_i, l_i)} = 1.$$

By Corollary 9.11, this implies that

$$\sum_{Y' \in p_*^{-1}(Y)} q^{\underline{\mathrm{wt}}(Y')} \Big|_{q_1 = \cdots = q_n = \xi, \, q_0 = \xi^2 q} = q^{\mathrm{wt}_0(Y)},$$

which proves Theorem 9.8.

9.2. Young walls and abacus of type D_n

We continue to work with the root system Δ of type D_n, and introduce some associated combinatorics. From now on, we return to the untransformed representation of the type D_n Young wall pattern introduced in 5.2.

Recalling the Young wall rules (YW1)-(YW4), it is clear that every $Y \in \mathcal{Z}_\Delta$ can be decomposed as $Y = Y_1 \sqcup Y_2$, where $Y_1 \in \mathcal{Z}_\Delta$ has full columns only, and $Y_2 \in \mathcal{Z}_\Delta$ has all its columns ending in a half-block. These conditions define two subsets $\mathcal{Z}_\Delta^f, \mathcal{Z}_\Delta^h \subset \mathcal{Z}_\Delta$. Because of the Young wall rules, the pair (Y_1, Y_2) uniquely reconstructs Y, so we get a bijection

(9.2) $$\mathcal{Z}_\Delta \longleftrightarrow \mathcal{Z}_\Delta^f \times \mathcal{Z}_\Delta^h.$$

Given a Young wall $Y \in \mathcal{Z}_\Delta$ of type D_n, let λ_k denote the number of blocks (full or half blocks both contributing 1) in the k-th vertical column. By the rules of Young walls, the resulting positive integers $\{\lambda_1, \ldots, \lambda_r\}$ form a partition $\lambda = \lambda(Y)$ of weight equal to the total weight $|Y|$, with the additional property that its parts λ_k are distinct except when $\lambda_k \equiv 0 \mod (n-1)$. Corresponding to the decomposition (9.2), we get a decomposition $\lambda(Y) = \mu(Y) \sqcup \nu(Y)$. In $\mu(Y)$, no part is congruent to 0 modulo $(n-1)$, and there are no repetitions; all parts in $\nu(Y)$ are congruent to 0 modulo $(n-1)$ and repetitions are allowed. Note that the pair $(\mu(Y), \nu(Y))$ does almost, but not quite, encode Y, because of the ambiguity in the labels of half-blocks on top of non-complete columns.

We now introduce the type[1] D_n abacus, following [39, 41]. This abacus is the arrangement of positive integers, called positions, in the following pattern.

---

[1] Once again, we should call it type $\tilde{D}_n^{(1)}$, but we simplify for ease of notation.

$$
\begin{array}{cccccccc}
1 & \ldots & n-2 & n-1 & n & \ldots & 2n-3 & 2n-2 \\
2n-1 & \ldots & 3n-4 & 3n-3 & 3n-2 & \ldots & 4n-5 & 4n-4 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots
\end{array}
$$

For any integer $1 \leq k \leq 2n-2$, the set of positions in the k-th column of the abacus is the k-th runner, denoted R_k. As in type A, positions on the runners are occupied by beads. For $k \not\equiv 0 \bmod (n-1)$, the runners R_k can only contain normal (uncolored) beads, with each position occupied by at most one bead. On the runners R_{n-1} and R_{2n-2}, the beads are colored white and black. An arbitrary number of white or black beads can be put on each such position, but each position can only contain beads of the same color.

Given a type D_n Young wall $Y \in \mathcal{Z}_\Delta$, let $\lambda = \mu \sqcup \nu$ be the corresponding partition with its decomposition. For each nonzero part ν_k of ν, set

$$
n_k = \#\{1 \leq j \leq l(\mu) \mid \mu_j < \nu_k\}
$$

to be the number of full columns shorter than a given non-full column. The abacus configuration of the Young wall Y is defined to be the set of beads placed at positions $\lambda_1, \ldots, \lambda_r$. The beads at positions $\lambda_k = \mu_j$ are uncolored; the color of the bead at position $\lambda_k = \nu_l$ corresponding to a column C of Y is

$$
\begin{cases}
\text{white,} & \text{if the block at the top of } C \text{ is } \triangle \text{ and } n_l \text{ is even;} \\
\text{white,} & \text{if the block at the top of } C \text{ is } \triangledown \text{ and } n_l \text{ is odd;} \\
\text{black,} & \text{if the block at the top of } C \text{ is } \triangledown \text{ and } n_l \text{ is even;} \\
\text{black,} & \text{if the block at the top of } C \text{ is } \triangle \text{ and } n_l \text{ is odd.}
\end{cases}
$$

One can check that the abacus rules are satisfied, that all abacus configurations satisfying the above rules, with finitely many uncolored, black and white beads, can arise, and that the Young wall Y is uniquely determined by its abacus configuration.

EXAMPLE 9.1. The abacus configuration associated to the Young wall Y_6 of Example 8.3 is

R_1	R_2	R_3	R_4	R_5	R_6
1	2	3	4	5	⑥
⑦	⑧	9	⑩	⑪	⑫
13	14	15	16	17	18
$\vdots$			$\vdots$		

9.3. Core Young walls and their abacus representation

In parallel with the type A story, we now introduce the combinatorics of core Young walls of type D_n, and the corresponding abacus moves. On the Young wall side, define a *bar* to be a connected set of blocks and half-blocks, with each half-block occuring once and each block occuring twice. A Young wall $Y \in \mathcal{Z}_\Delta$ will be called a

core Young wall, if no bar can be removed from it without violating the Young wall rules. For an example of bar removal, see [39, Example 5.1(2)]. Let $\mathcal{C}_\Delta \subset \mathcal{Z}_\Delta$ denote the set of all core Young walls. The following statement is the type D analogue of the discussion of 4.4.

PROPOSITION 9.2. ([39, 41]) *Given a Young wall* $Y \in \mathcal{Z}_\Delta$, *any complete sequence of bar removals through Young walls results in the same core* $\mathrm{core}(Y) \in \mathcal{C}_\Delta$, *defining a map of sets*

$$\mathrm{core} \colon \mathcal{Z}_\Delta \to \mathcal{C}_\Delta.$$

The process can be described on the abacus, respects the decomposition (9.2), *and results in a bijection*

$$(9.3) \qquad \mathcal{Z}_\Delta \longleftrightarrow \mathcal{P}^{n+1} \times \mathcal{C}_\Delta$$

where $\mathcal{P}$ *is the set of ordinary partitions. Finally, there is also a bijection*

$$(9.4) \qquad \mathcal{C}_\Delta \longleftrightarrow \mathbb{Z}^n.$$

PROOF. Decompose Y into a pair of Young walls (Y_1, Y_2) as above. Let us first consider Y_1. On the corresponding runners R_k, $k \not\equiv 0 \mod (n-1)$, the following steps correspond to bar removals [39, Lemma 5.2].

(B1) If b is a bead at position $s > 2n - 2$, and there is no bead at position $(s - 2n + 2)$, then move b one position up and switch the color of the beads at positions k with $k \equiv 0 \mod (n-1)$, $s - 2n + 2 < k < s$.

(B2) If b and b' are beads at position s and $2n - 2 - s$ ($1 \leq s \leq n - 2$) respectively, then remove b and b' and switch the color of the beads at positions $k \equiv 0 \mod (n-1)$, $s < k < 2n - 2 - s$.

Performing these steps as long as possible results in a configuration of beads on the runners R_k with $k \not\equiv 0 \mod (n-1)$ with no gaps from above, and for $1 \leq s \leq n-2$, beads on only one of R_s, R_{2n-2-s}. This final configuration can be uniquely described by an ordered set of integers $\{z_1, \ldots, z_{n-2}\}$, z_s being the number of beads on R_s minus the number of beads on R_{2n-2-s} [41, Remark 3.10(2)]. In the correspondence (9.4) this gives $\mathbb{Z}^{n-2}$. It turns out that the reduction steps in this part of the algorithm can be encoded by an $(n-2)$-tuple of ordinary partitions, with the summed weight of these partitions equal to the number of bars removed [39, Theorem 5.11(2)].

Let us turn to Y_2, represented on the runners R_k, $k \equiv 0 \mod (n-1)$. On these runners, the following steps correspond to bar removals [41, Sections 3.2 and 3.3].

(B3) Let b be a bead at position $s \geq 2n - 2$. If there is no bead at position $(s - n + 1)$, and the beads at position $(s - 2n + 2)$ are of the same color as b, then shift b up to position $(s - 2n + 2)$.

(B4) If b and b' are beads at position $s \geq n - 1$, then move them up to position $(s - n + 1)$. If $s - n + 1 > 0$ and this position already contains beads, then b and b' take that same color.

During these steps, there is a boundary condition: there is an imaginary position 0 in the rightmost column, which is considered to contain unvisible white beads; placing a bead there means that this bead disappears from the abacus. It turns out that the reduction steps in this part of the algorithm can be described by a triple of ordinary partitions, again with the summed weight of these partitions equal to the number of bars removed [41, Proposition 3.6]. On the other hand, the final result can be encoded by a pair of 2-core partitions which appeared in Section 4.4.

The different bar removal steps (B1)-(B4) construct the map c algorithmically and uniquely. The stated facts about parameterizing the steps prove the existence of the bijection (9.3). To complete the proof of (9.4), we only need to remark further that the set of 2-core partitions, in our language A_1-core partitions, is in bijection with the set of integers by bijection (4.11) in Section 4.4 (see also [41, Remark 3.10(2)]). This gives the remaining $\mathbb{Z}^2$ factor in the bijection (9.4). $\qquad\square$

We next determine the multi-weight of a Young wall Y in terms of the bijections (9.3)-(9.4). The quotient part is easy: the multi-weight of each bar is $(1, 1, 2, \ldots, 2, 1, 1)$, so in complete analogy with the type A situation, the contribution of the $(n + 1)$-tuple of partitions to the multi-weight is easy to compute. Turning to cores, under the bijection $\mathcal{C}_\Delta \leftrightarrow \mathbb{Z}^n$, the total weight of a core Young wall $Y \in \mathcal{C}_\Delta$ corresponding to $(z_1, \ldots, z_n) \in \mathbb{Z}^n$ is calculated in [41, Remark 3.10]:

$$(9.5) \qquad |Y| = \frac{1}{2} \sum_{i=1}^{n-2} \left((2n - 2)z_i^2 - (2n - 2i - 2)z_i \right) + (n - 1) \sum_{i=n-1}^{n} \left(2z_i^2 + z_i \right).$$

A refinement of this formula calculates the multi-weight of Y.

THEOREM 9.3. *Let* $q = q_0 q_1 q_2^2 \ldots q_{n-2}^2 q_{n-1} q_n$, *corresponding to a single bar.*

(1) *Composing the bijection* (9.4) *with an appropriate* $\mathbb{Z}$*-change of coordinates in the lattice* $\mathbb{Z}^n$, *the multi-weight of a core Young wall* $Y \in \mathcal{C}_\Delta$ *corresponding to an element* $\overline{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ *can be expressed as*

$$q_1^{m_1} \cdot \ldots \cdot q_n^{m_n} (q^{1/2})^{\overline{m}^\top \cdot C \cdot \overline{m}},$$

where C *is the Cartan matrix of type* D_n.

(2) *The multi-weight generating series*

$$Z_\Delta(q_0, \ldots, q_n) = \sum_{Y \in \mathcal{Z}_\Delta} \underline{q}^{\underline{wt}(Y)}$$

of Young walls for Δ *of type* D_n *can be written as*

$$Z_\Delta(q_0, \ldots, q_n) = \frac{\displaystyle\sum_{\overline{m}=(m_1,\ldots,m_n)\in\mathbb{Z}^n}^{\infty} q_1^{m_1} \cdot \ldots \cdot q_n^{m_n} (q^{1/2})^{\overline{m}^\top \cdot C \cdot \overline{m}}}{\displaystyle\prod_{m=1}^{\infty} (1 - q^m)^{n+1}}.$$

(3) *The following identity is satisfied between the coordinates $(m_1, \ldots, m_n)$ and $(z_1, \ldots, z_n)$ on $\mathbb{Z}^n$:*

$$\sum_{i=1}^{n} m_i = -\sum_{i=1}^{n-2}(n-1-i)z_i - (n-1)c(z_{n-1}+z_n) - (n-1)b.$$

Here $z_1 + \cdots + z_{n-2} = 2a - b$ for integers $a \in \mathbb{Z}$, $b \in \{0,1\}$, and $c = 2b-1 \in \{-1,1\}$.

The coordinate change $(z_1, \ldots, z_n) \mapsto (m_1, \ldots, m_n)$ and the multiweight formula of (1), as well as (3), follow from somewhat involved but routine calculations. (2) clearly follows from (1) and the preceeding discussion.

Let us write $z_I = \sum_{i \in I} z_i$ for $I \subseteq \{1, \ldots, n-2\}$. Each such number decomposes uniquely as $z_I = 2a_I - b_I$, where $a_I \in \mathbb{Z}$ and $b_I \in \{0,1\}$. Let us introduce also $c_I = 2b_I - 1 \in \{-1,1\}$. We will make use of the relations

$$a_I = \sum_{i \in I} a_i - \sum_{i_1 \in I, i_2 \in I \setminus \{i_1\}} b_{i_1} b_{i_2} + \sum_{i_1 \in I, i_2 \in I \setminus \{i_1\}, i_3 \in I \setminus \{i_1, i_2\}} 2b_{i_1} b_{i_2} b_{i_3} - \ldots \,,$$

$$b_I = \sum_{i \in I} b_i - \sum_{i_1 \in I, i_2 \in I \setminus \{i_1\}} 2b_{i_1} b_{i_2} + \sum_{i_1 \in I, i_2 \in I \setminus \{i_1\}, i_3 \in I \setminus \{i_1, i_2\}} 4b_{i_1} b_{i_2} b_{i_3} - \ldots \,.$$

To simplify notations let us introduce

$$r_I := a_I - \sum_{i \in I} a_i = -\sum_{i_1 \in I, i_2 \in I \setminus \{i_1\}} b_{i_1} b_{i_2} + \sum_{i_1 \in I, i_2 \in I \setminus \{i_1\}, i_3 \in I \setminus \{i_1, i_2\}} 2b_{i_1} b_{i_2} b_{i_3} - \ldots \,.$$

Using these notations the colored refinement of the weight formula (9.5) is the following.

LEMMA 9.4. *Given a core Young wall $Y \in \mathcal{C}_\Delta$ corresponding to $(z_i) \in \mathbb{Z}^n$ in the bijection of (9.4), its content is given by the formula*

$$\underline{q}^{\underline{wt}(Y)} = q_1^{-\sum_{i=1}^{n-2} b_i} q_2^{-2a_1 - \sum_{i=2}^{n-2} b_i} \cdots q_{n-2}^{-\sum_{i=1}^{n-3} 2a_i - b_{n-2}} (q_0 q_1^{-1} q_{n-1} q_n)^{-\sum_{i=1}^{n-2} a_i} (q_0 q_1^{-1})^{a_{1\ldots n-2}}$$

$$\cdot q^{\frac{1}{2} \sum_{i=1}^{n-2}(z_i^2 + b_i) + z_{n-1}^2 + z_n^2}$$

$$\cdot (q^{b_{1\ldots n-2}}(q_1^{-1} \cdots q_{n-2}^{-1} q_{n-1}^{-1})^{c_{1\ldots n-2}})^{z_{n-1}} (q^{b_{1\ldots n-2}}(q_1^{-1} \cdots q_{n-2}^{-1} q_n^{-1})^{c_{1\ldots n-2}})^{z_n}.$$

PROOF. When forgetting the coloring the lemma obviously gives back (9.5). Notice also that $z_i^2 + b_i = 4a_i^2 - 4a_i b_i + 2b_i$ is always an even number, so the exponents are always integers. Later we will show that the expression is a multivariable theta function.

The total weight of a core basically measures the area of its Young wall. So the exponents can be at most quadratic in the parameters $\{z_i\}_{1 \le i \le n}$. Hence, it is enough to check that the formula is correct in two cases:

(1) when any of the z_i's is set to a given number and the others are fixed to 0; and

(2) when all of the parameters are fixed to 0 except for an arbitrary pair z_i and z_j, $i \neq j$.

First, consider that $z_i \neq 0$ for a fixed i, and $z_j = 0$ in case $j \neq i$.

(a) When $1 \leq i \leq n - 2$, then the colored weight of the corresponding core Young wall is

$$(q_1 \ldots q_i)^{-b_i}(q_{i+1}^2 \ldots q_{n-2}^2 q_{n-1} q_n)^{-a_i} q^{2a_i^2 - 2a_i b_i + b_i} \ .$$

(b) When $i \in \{n - 1, n\}$, then the associated core Young wall has colored weight

$$q^{z_i^2}(q_1 q_2 \ldots q_{n-2} q_i)^{z_i} \ .$$

Both of these follow from (9.5) and its proof in [41] by taking into account the colors of the blocks in the pattern.

Second, assume that z_i and z_j are nonzero, but everything else is zero. Then the total weight is not the product of the two individual weights, but some correction term has to be introduced. The particular cases are:

(a) $1 \leq i, j \leq n - 2$. There can only be a difference in the numbers of q_0's and q_1's which comes from the fact that in the first row there are only half blocks with 0's in the odd columns and 1's in the even columns. Exactly $-r_{ij}$ blocks change color from 0 to 1 when both z_i and z_j are nonzero compared to when one of them is zero. In general, this gives the correction term $(q_0 q_1^{-1})^{r_{1\ldots n-2}} = (q_0 q_1^{-1})^{a_{1\ldots n-2} - \sum_{i=1}^{n-2} a_i}$.

(b) $1 \leq i \leq n - 2$, $j \in \{n - 1, n\}$. For the same reason as in the previous case the parity of z_i modifies the colored weight of the contribution of z_j, but not the total weight of it. If z_i is even, then the linear term of the contribution of z_j is $q_1 q_2 \ldots q_{n-2} q_j$. In the odd case it is $q_0 q_2 \ldots q_{n-2} q_{\kappa(j)}$. This is encoded in the correction term $(q^{b_{1\ldots n-2}}(q_1^{-1} \ldots q_{n-2}^{-1} q_j^{-1})^{c_{1\ldots n-2}})^{z_j}$.

(c) $i = n - 1$, $j = n$. z_{n-1} and z_n count into the total colored weight completely independently, so no correction term is needed.

Putting everything together gives the claimed formula for the colored weight of an arbitrary core Young wall.

$\square$

Now we turn to the proof of Theorem 9.3. After recollecting the terms in the formula of Lemma 9.4 it becomes

$$q_1^{-b_{1\ldots n-2} - c_{1\ldots n-2}(z_{n-1}+z_n)} \prod_{i=2}^{n-2} q_i^{-2a_{1\ldots i-1} + c_{1\ldots i-1} b_{i\ldots n-2} - c_{1\ldots n-2}(z_{n-1}+z_n)}$$

$$\cdot q_{n-1}^{-a_{1\ldots n-2} - c_{1\ldots n-2} z_{n-1}} q_n^{-a_{1\ldots n-2} - c_{1\ldots n-2} z_n}$$

$$\cdot q^{\sum_{i=1}^{n-2}(2a_i^2 - 2a_i b_i + b_i) + b_{1\ldots n-2} z_{n-1} + z_{n-1}^2 + b_{1\ldots n-2} z_n + z_n^2 + r_{1\ldots n-2}}$$

Let us define the following series of integers:

$$
\begin{aligned}
m_1 &= -b_{1...n-2} - c_{1...n-2}(z_{n-1} + z_n) \,, \\
m_2 &= -2a_1 + c_1 b_{2...n-2} - c_{1...n-2}(z_{n-1} + z_n) \,, \\
&\;\;\vdots \\
m_{n-2} &= -2a_{1...n-3} + c_{1...n-3}b_{n-2} - c_{1...n-2}(z_{n-1} + z_n) \,, \\
m_{n-1} &= -a_{1...n-2} - c_{1...n-2}z_{n-1} \,, \\
m_n &= -a_{1...n-2} - c_{1...n-2}z_n \,.
\end{aligned}
$$

It is an easy and enlightening task to verify that the map

$$
\mathbb{Z}^n \to \mathbb{Z}^n, \quad (z_1, \ldots, z_n) \mapsto (m_1, \ldots, m_n)
$$

is a bijection, which is left to the reader.

PROOF OF THEOREM 9.3. (1): One has to check that

$$
\sum_{i=1}^{n} m_i^2 - m_1 m_2 - m_2 m_3 - \cdots - m_{n-2}(m_{n-1} + m_n) =
$$

$$
= \sum_{i=1}^{n-2}(2a_i^2 - 2a_i b_i + b_i) + b_{1...n-2}z_{n-1} + z_{n-1}^2 + b_{1...n-2}z_n + z_n^2 + r_{1...n-2} \,.
$$

The terms containing z_{n-1} or z_n on the left hand side are

$$
(n-2)(z_{n-1} + z_n)^2 + z_{n-1}^2 + z_n^2 - (n-3)(z_{n-1} + z_n)^2 - z_{n-1}^2 - z_n^2 - 2z_{n-1}z_n
$$

$$
+ \left(2b_{1...n-2} + \sum_{i=1}^{n-3} 2(2a_{1...i} - c_{1...i}b_{i+1...n-2}) + 2a_{1...n-2} \right) c_{1...n-2}(z_{n-1} + z_n)
$$

$$
- \left(b_{1...n-2} + \sum_{i=1}^{n-3} 2(2a_{1...i} - c_{1...i}b_{i+1...n-2}) + 2a_{1...n-2} \right) c_{1...n-2}(z_{n-1} + z_n)
$$

$$
= b_{1...n-2}z_{n-1} + z_{n-1}^2 + b_{1...n-2}z_n + z_n^2 \,,
$$

since $b_{1...n-2}c_{1...n-2} = b_{1...n-2}$.

The terms containing neither z_{n-1} nor z_n on the left hand side are

$$
b_{1...n-2} + \sum_{i=1}^{n-3}(2a_{1...i} - c_{1...i}b_{i+1...n-2})^2 + 2a_{1...n-2}^2 - b_{1...n-2}(2a_1 - c_1 b_{2...n-2})
$$

$$
- \sum_{i=1}^{n-4}(2a_{1...i} - c_{1...i}b_{i+1...n-2})(2a_{1...i+1} - c_{1...i+1}b_{i+2...n-2})
$$

$$
- 2(2a_{1...n-3} - c_{1...n-3}b_{n-2})a_{1...n-2} \,.
$$

LEMMA 9.5.

$$
2a_{1...i} - c_{1...i}b_{i+1...n-2} = \sum_{j=1}^{i}(2a_j - b_j) + b_{1...n-2} \,,
$$

PROOF.

$$2a_{1...i} - c_{1...i}b_{i+1...n-2}$$

$$= 2a_{1...i-1} + 2a_i - 2b_{1...i-1}b_i + c_{1...i-1}c_ib_{i+1...n-2}$$

$$= 2a_{1...i-1} + 2a_i - 2b_{1...i-1}b_i + 2c_{1...i-1}b_ib_{i+1...n-2} - c_{1...i-1}b_{i+1...n-2}$$

$$= 2a_{1...i-1} + 2a_i - b_i - c_{1...i-1}(b_{i+1...n-2} + b_i - 2b_ib_{i+1...n-2})$$

$$= 2a_{1...i-1} - c_{1...i-1}b_{i...n-2} + 2a_i - b_i \ ,$$

and then use induction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Applying Lemma 9.5 and the last intermediate expression in its proof to the terms considered above, they simplify to

$$b_{1...n-2} + \sum_{i=1}^{n-3}(2a_{1...i} - c_{1...i}b_{i+1...n-2})^2 + 2a_{1...n-2}^2 - b_{1...n-2}(2a_1 - c_1b_{2...n-2})$$

$$- \sum_{i=1}^{n-4}(2a_{1...i} - c_{1...i}b_{i+1...n-2})(2a_{1...i} - c_{1...i}b_{i+1...n-2} + 2a_{i+1} - b_{i+1})$$

$$-2(2a_{1...n-3} - c_{1...n-3}b_{n-2})a_{1...n-2}$$

$$= b_{1...n-2} + (2a_{1...n-3} - c_{1...n-3}b_{n-2})^2 + 2a_{1...n-2}^2 - b_{1...n-2}(2a_1 - c_1b_{2...n-2})$$

$$- \sum_{i=1}^{n-4}(2a_{1...i} - c_{1...i}b_{i+1...n-2})(2a_{i+1} - b_{i+1}) - 2(2a_{1...n-3} - c_{1...n-3}b_{n-2})a_{1...n-2}$$

$$= b_{1...n-2} + (2a_{1...n-3} - c_{1...n-3}b_{n-2})(2a_{1...n-3} - c_{1...n-3}b_{n-2} - 2a_{1...n-2})$$

$$+2a_{1...n-2}^2 - b_{1...n-2}(2a_1 - c_1b_{2...n-2}) - \sum_{i=1}^{n-4}(2a_{1...i} - c_{1...i}b_{i+1...n-2})(2a_{i+1} - b_{i+1})$$

$$= b_{1...n-2} + 2a_{1...n-2}^2 - b_{1...n-2}(2a_1 - c_1b_{2...n-2})$$

$$- \sum_{i=1}^{n-3}(2a_{1...i} - c_{1...i}b_{i+1...n-2})(2a_{i+1} - b_{i+1})$$

$$= 2a_{1...n-2}^2 - b_{1...n-2}(2a_1 - b_1) - \sum_{i=1}^{n-3}\left(\sum_{j=1}^{i}(2a_j - b_j) + b_{1...n-2}\right)(2a_{i+1} - b_{i+1}) \ .$$

Let us denote this expression temporarily as s_{n-2}. Taking into account that

$$a_{1...n-2} = a_{1...n-3} + a_{n-2} - b_{1...n-3}b_{n-2} \ ,$$

$$b_{1...n-2} = b_{1...n-3} + b_{n-2} - 2b_{1...n-3}b_{n-2} \ ,$$

s_{n-2} can be rewritten as

$$2a_{1\ldots n-3}^2 + 2a_{n-2}^2 + 2b_{1\ldots n-3}b_{n-2} + 4a_{1\ldots n-3}a_{n-2}$$

$$-4a_{1\ldots n-3}b_{1\ldots n-3}b_{n-2} - 4a_{n-2}b_{1\ldots n-3}b_{n-2}$$

$$-(b_{1\ldots n-3} + b_{n-2} - 2b_{1\ldots n-3}b_{n-2})(2a_1 - b_1)$$

$$-\sum_{i=1}^{n-4}\left(\sum_{j=1}^{i}(2a_j - b_j) + b_{1\ldots n-3}\right)(2a_{i+1} - b_{i+1})$$

$$-\sum_{i=1}^{n-3}(b_{n-2} - 2b_{1\ldots n-3}b_{n-2})(2a_{i+1} - b_{i+1})$$

$$-\left(\sum_{j=1}^{n-3}(2a_j - b_j) + b_{1\ldots n-3}\right)(2a_{n-2} - b_{n-2})$$

$$= s_{n-3} + 2a_{n-2}^2 + 2b_{1\ldots n-3}b_{n-2} + 4a_{1\ldots n-3}a_{n-2}$$

$$-4a_{1\ldots n-3}b_{1\ldots n-3}b_{n-2} - 4a_{n-2}b_{1\ldots n-3}b_{n-2}$$

$$-(b_{n-2} - 2b_{1\ldots n-3}b_{n-2})\left(\sum_{i=1}^{n-2}2a_i - b_i\right) - \left(\sum_{j=1}^{n-3}(2a_j - b_j) + b_{1\ldots n-3}\right)(2a_{n-2} - b_{n-2})$$

$$= s_{n-3} + 2a_{n-2}^2 - 2a_{n-2}b_{n-2} + b_{n-2} + 2a_{n-2}\left(2a_{1\ldots n-3} - \sum_{j=1}^{n-3}(2a_j - b_j)\right)$$

$$+b_{1\ldots n-3}b_{n-2} - 2a_{n-2}b_{1\ldots n-3}$$

$$= s_{n-3} + 2a_{n-2}^2 - 2a_{n-2}b_{n-2} + b_{n-2} - b_{1\ldots n-3}b_{n-2} \; ,$$

where at the last equality the identity $\sum_{j=1}^{n-3}(2a_j - b_j) = z_{1\ldots n-3} = 2a_{1\ldots n-3} - b_{1\ldots n-3}$ was used.

It can be checked, that $s_1 = 2a_1^2 - 2a_1 b_1 + b_1$, so induction shows that

$$s_{n-2} = \sum_{i=1}^{n-2}(2a_i^2 - 2a_i b_i + b_i + b_{1\ldots i-1}b_i) \; .$$

It remains to show that

$$\sum_{i=2}^{n-2} b_{1\ldots i-1}b_i = r_{1\ldots n-2} \; ,$$

which requires another induction argument, and is left to the reader.

(3): Apply Lemma 9.5 on

$$\sum_{i=1}^{n} m_i = -b_{1\ldots n-2} - \sum_{i=1}^{n-2}(2a_{1\ldots i-1} - c_{1\ldots i-1}b_{i\ldots n-2}) - 2a_{1\ldots n-2} - (n-1)c_{1\ldots n-2}(z_{n-1} + z_n).$$

$\square$

9.4. 0-generated Young walls and their abacus representations

In this section, we characterize the abacus configurations corresponding to Young walls in the sets $\mathcal{Z}_\Delta^0$ and $\mathcal{Z}_\Delta^1$ defined in 8.2.

Recall conditions (R1)–(R3) on Young walls $Y \in \mathcal{Z}_\Delta$ from 8.2. Recall also that a Young wall corresponds uniquely to an abacus configuration, where the beads are placed at the positions $\lambda_1, \ldots, \lambda_r$. Finally recall that the quantity n_k denotes the number of full columns shorter than a given non-full column of height λ_k.

LEMMA 9.6. *Conditions (R1)-(R2) on Young walls are equivalent to the following conditions for an abacus configuration.*

(D1) *In each row, the rightmost bead is always on the $(2n-2)$-nd ruler, and either all the beads of the row are at this position, or the number of beads in this position is odd.*

(D2) *If a row ends with a white (resp., black) bead corresponding to a column of height λ_k, then $k + n_k$ must be odd (resp. even). If several beads are placed on this position, which is allowed since it is on the ruler R_{2n-2}, then this condition refers to the smallest possible k.*

(D3) *The total number of beads in the whole abacus is even, or the total number of beads on the rulers $R_1, \ldots, R_{n-1}$ in the first row is $n - 2$.*

(D4) *The beads on the rulers $R_n \ldots, R_{2n-3}$ are pushed to the right as much as possible in each row. In any given row, the positions on the rulers $R_1, \ldots, R_{n-1}$ are empty unless all the positions on the rulers $R_n, \ldots, R_{2n-2}$ are filled.*

(D5) *The beads on the rulers $R_1 \ldots R_{n-1}$ in any given row are either all on the ruler R_{n-1}, or on the rules $R_1 \ldots R_{n-2}$, and pushed to the right as much as possible.*

Condition (R3) is equivalent to the following condition.

(D6) *Let s be the total number of beads on the rulers $R_1, \ldots, R_{n-1}$ in any given row.*
    (a) *If $s > n - 2$, then all these beads are on R_{n-1}.*
    (b) *If $s \leq n - 2$, then all these beads are on the rulers $R_1, \ldots, R_{n-2}$, pushed to the right.*

Thus 0-generated Young walls $Y \in \mathcal{Z}_\Delta^1$ correspond to abacus configurations satisfying (D1)-(D5), whereas distinguished 0-generated Young walls $Y \in \mathcal{Z}_\Delta^0$ correspond to those satisfying (D6) also.

PROOF. Two kinds of salient blocks can appear in a Young wall that satisfies (R1)-(R2):

- label 0 half-blocks,
- and salient block-pairs of label $n - 1$ or n.

As in the type A case a salient block corresponds to the first bead in a consecutive series of beads in the abacus. More precisely, if there are several columns of height λ_k, or equivalently, if there are several beads placed on the position λ_k, then the salient block corresponds to that column of height λ_k which has the smallest possible index k among these.

The label 0 blocks always correspond to positions which are on the ruler R_{2n-2}. In the odd columns of the type D pattern they are of the shape $\triangledown$ while in the even columns they are of the shape $\triangle$. Condition (D2) encodes the fact, that the salient blocks of label 0 are upper triangles in odd columns and lower triangles in even columns, and that the color of the beads corresponding to them is also affected by the parity of the appropriate n_k.

If there is a salient block of label 0, then some columns of the same height, let's say, λ_k, can follow it. If the first column after them has height $\lambda_k - 1$ then on its top there is again a salient block of label 0. This block can only have the opposite orientation than the aforementioned salient block, hence the number of columns of height λ_k in this case can only be odd. This gives condition (D1).

Condition (D3) follows again from the absence of label 1 salient blocks. To see this recall that in the bottom row of the type D pattern there are half blocks which have label 0 in the odd columns and have label 1 in the even columns. Since there are no salient blocks of label 1 in Y, this total number of columns can only be odd if the last column has height 1, the column to the left of it has height 2, and so on. This is can only happen when in the bottom row $s = n - 2$.

The fact that there is no salient block of label $2, \ldots, n - 2$ implies that for each bead on the rulers $R_1, \ldots, R_{2n-1}$ there has to be a block placed on its right. The only exception is the ruler R_{n-2}. There cannot be any bead on this ruler, except when there is a salient block pair of label $n - 1/n$ which corresponds to a hole on R_{n-1}. These considerations imply conditions (D4) and (D5).

Condition (D6) corresponds directly to property (R3). $\square$

Given $Y \in \mathcal{Z}_\Delta^0$, let t_i denote the total number of beads in the i-th row of its abacus representation, and l_i the number of beads in the rightmost position of the i-th row. We obtain a sequence of pairs $(t_i, l_i)_{i \in \mathbb{Z}_+}$, only finitely many of which do not equal $(0, 0)$.

COROLLARY 9.7. *Given $Y \in \mathcal{Z}_\Delta^0$, the associated sequence of pairs $(t_i, l_i)_{i \in \mathbb{Z}_+}$ satisfies the following conditions.*

(1) For all i, $0 \le l_i \le t_i$.
(2) For all i, if $t_i > 0$, then either $l_i = t_i$ is even, or l_i is odd.
(3) Either $\sum_i t_i$ is even, or $t_1 - l_1 = 2n - 4$.

Conversely, any sequence $(t_i, l_i)_{i \in \mathbb{Z}_+}$ satisfying these conditions arises as a sequence associated to at least one Young wall $Y \in \mathcal{Z}_\Delta^0$. More precisely, the number of different

Young walls $Y \in \mathcal{Z}_\Delta^0$ corresponding to any given sequence is 2^m, where m is the number of indices i such that $t_i - l_i > 2n - 2$. All Young walls Y corresponding to a single sequence have the same multi-weight, when the weights for labels $n - 1$ and n are counted together.

PROOF. Condition (1) is clear from the definition of (t_i, l_i). Condition (2) follows from (D1) above. Condition (3) is equivalent to condition (D3).

Conversely, given a sequence $(t_i, l_i)_{i \in \mathbb{Z}_+}$ satisfying conditions (1)-(3), we can reconstruct a corresponding $Y \in \mathcal{Z}_\Delta^0$ in its abacus representation as follows. On the i-th row, l_i beads have to be put on the last position; (D1) is satisfied because of (1). They are white if $\sum_{j<i} t_j + \sum_{j>i, t_j \not\equiv 0 \bmod n-1} t_j \equiv 1 \bmod 2$, and black otherwise; this is just a reformulation of (D2). (D3) is satisfied because of (3). At most one bead can be put on each ruler between R_n and R_{2n-3}, pushed to the right as much as possible; this is (D4). If $t_i - l_i \leq 2n - 2$, then the rest of the beads fill up the rulers between R_1 and R_{n-2}, pushed to the right. If $t_i - l_i > 2n - 2$, then there are no beads in this row on the rulers between R_1 and R_{n-2}; instead, the remaining beads are all placed on the $(n-1)$-st ruler, and they can be either white or black. These rules reconstruct a configuration satisfying (D5)-(D6), and give the stated ambiguity in the reconstruction.   □

9.5. The generating series of distinguished 0-generated walls

In light of Theorem 8.14, in order to complete the proof of our main Theorem 3.5 for type D, we need the following combinatorial result, the precise analogue of Proposition 4.19 in type A.

THEOREM 9.8. *Let Δ be of type D_n, and let ξ be a primitive $(2n-1)$-st root of unity. Then the generating series of the set $\mathcal{Z}_\Delta^0$ of distinguished 0-generated Young walls is given in terms of the generating function of all Young walls by the following substitution:*

$$\sum_{Y \in \mathcal{Z}_\Delta^0} q^{\mathrm{wt}_0(Y)} = Z_\Delta(q_0, \dots, q_n)\Big|_{q_0 = \xi^2 q, \, q_1 = \cdots = q_n = \xi}.$$

In analogy once again with the type A proof, the following is the key construction in our proof of this result. There is a combinatorial map

$$p \colon \mathcal{Z}_\Delta \to \mathcal{Z}_\Delta^0$$

defined as follows. For an arbitrary Young wall $Y \in \mathcal{Z}_\Delta$ we take the Young wall $Y_1 \in \mathcal{Z}_\Delta'$ which contains Y and which is minimal with this property with respect to containment. By Lemma 8.1 Y_1 is unique. We let $p(Y) = \mathrm{red}(Y_1)$, where $\mathrm{red} \colon \mathcal{Z}_\Delta' \to \mathcal{Z}_\Delta^0$ is the map defined in Lemma 8.2. We remark that in fact $p(Y)$ is the unique element in $\mathcal{Z}_\Delta^0$ which has exactly the same set of label 0 blocks as Y.

PROPOSITION 9.9. *On the abacus representation of Young walls, the map* $p\colon \mathcal{Z}_\Delta \to \mathcal{Z}_\Delta^0$ *corresponds to the following steps:*

(1) *If a row ends with a white (resp., black) bead on R_{2n-2} corresponding to a column of height λ_k, and $k + n_k$ is even (resp. odd), then one bead should be moved to the next position, which is the leftmost in the next row. This is applied also on the zeroth position of the abacus, where we assume that there are infinitely many beads. This corresponds to the appearance of a new column in the Young wall represented by the abacus.*

(2) *Every bead on the rulers $R_1, \ldots, R_{2n-4}$ is moved to right as much as possible according to the abacus rules.*

(3) *If there is at least one bead on the rulers $R_1, \ldots, R_{2n-3}$ after performing Step 1 on the previous row, and the number of beads on R_{2n-2} is even, then one more bead is moved onto R_{2n-2}. If there were beads on R_{2n-2} already, then this step does not change the parity of $k + n_k$ for the rightmost bead. If there were no beads on R_{2n-2} before, then this beads should take the appropriate color and it is possible to see that it can not be moved further with Step 1.*

(4) *Let s be the total number of beads on the rulers $R_1, \ldots, R_{n-1}$ after performing the Steps 1-3. If $s > n - 2$, then move all these beads on R_{n-1}. In this case some of these beads were here previously, so the color of the whole group of beads is already determined. If $s \leq n - 2$, then move them onto the rulers $R_1, \ldots, R_{n-2}$, each as right as possible.*

PROOF. Step 1 enforces condition (D2). It also enforces condition (D3) when applied to the minus first row. Step 2 enforces conditions (D4) and (D5), Step 3 enforces condition (D1), and finally Step 4 enforces condition (D6). $\qquad\square$

The fibers of the map p can be described as follows. Given a Young wall $Y \in \mathcal{Z}_\Delta^0$, we are allowed to move beads to the left and, occasionally, to the right, using the following rules.

(1) From the last position of the $i - th$ row only one (resp. zero) bead can be moved to the left if l_i is odd (resp., even).

(2) Every other bead is allowed to moved to the left in its row if the result is a valid abacus configuration.

(3) The leftmost bead in a row can be moved to the last position of the previous row. There it will take the color white if $\sum_{j<i} t_j + \sum_{j>i, t_j \not\equiv 0 \bmod n-1} t_j \equiv 1 \bmod 2$, and grey otherwise.

(4) If $t_i - l_i \leq 2n - 2$, then the beads to the left from the $n - 1$-st position are allowed to be moved to the right at most onto the ruler R_{n-1}.

(5) If $t_i - l_i \leq 2n - 2$, then any configuration, in which there is at least one bead at the $(n - 1)$-st position, has to be counted with multiplicity two.

Let us call the beads that can be moved according to these rules *movable*. For a row with data (t, l), let us also introduce the number $c(t, l)$, which is signed sum of the distance of the beads from the R_{n-1}-st ruler, where the sum runs over the movable beads, and the beads to the left of the R_{n-1}-st ruler are counted with negative sign, and the beads to the right of it are counted with positive sign. These numbers are listed in the table below:

	$l \equiv 0 \bmod 2$	$l \equiv 1 \bmod 2$
$0 \leq t - l \leq n - 2$	$\binom{n-1}{2} - \binom{n-1-t+l}{2}$	$\binom{n}{2} - \binom{n-1-t+l}{2}$
$n - 1 \leq t - l \leq 2n - 3$	$\binom{n-1}{2} - \binom{t-l-n+1}{2}$	$\binom{n}{2} - \binom{t-l-n+1}{2}$
$2n - 2 \leq t - l$	$\binom{n-1}{2}$	$\binom{n}{2}$

LEMMA 9.10. *The contribution of a row with data (t, l) to the total weight of the fiber is $\xi^{-c(t,l)}$.*

PROOF. If l is even but nonzero, then according to Corollary 9.7 $l = t$ and there isn't any movable bead.

If l is odd, then there is one movable bead on the R_{2n-2}-nd ruler. Assume first that $t \leq n - 1$. Then the expression

$$\sum_{n_1=0}^{2n-t+l-2} \sum_{n_2=0}^{n_1} \cdots \sum_{n_{t-l+1}=0}^{n_{t-l}} (\xi^{-1})^{n_1+\cdots+n_{t-l+1}} = \binom{2n-1}{t-l+1}_{\xi^{-1}}$$

counts once every preimage, in which there is at most one bead at the $n-1$-st position. Similarly

$$(\xi^{-1})^{n-t+l-1} \sum_{n_2=0}^{2n-t} \sum_{n_3=0}^{n_2} \cdots \sum_{n_{t-l+1}=0}^{n_{t-l}} (\xi^{-1})^{n_2+\cdots+n_{t-l+1}} = (\xi^{-1})^{n-t+l-1} \binom{2n-1}{t-l}_{\xi^{-1}}$$

counts once every preimage, in which there is at least one and at most two beads at the $n-1$-st position, since we moved one bead from the leftmost occupied position to the $n-1$-st position, and we fixed it there. The next term is given by

$$(\xi^{-1})^{(n-t+l-1)+(n-t+l)} \sum_{n_3=0}^{2n-t} \sum_{n_4=0}^{n_3} \cdots \sum_{n_{t-l+1}=0}^{n_{t-l}} (\xi^{-1})^{n_3+\cdots+n_{t-l+1}}$$

$$= (\xi^{-1})^{(n-t+l-1)+(n-t+l)} \binom{2n-1}{t-l-1}_{\xi^{-1}},$$

which counts once every preimage, in which there is at least two and at most three beads at the $n-1$-st ruler. Continuing in this fashion and summing up in the end we get

$$\binom{2n+1}{t-l+1}_{\xi^{-1}} + \sum_{i=1}^{t-l+1} \xi^{-\sum_{j=n-t+l-1}^{n-t+l-2+i} j} \binom{2n-1}{t-l+1-i}_{\xi^{-1}}.$$

It can be checked that $\binom{2n-1}{k}_{\xi^{-1}} = 0$ unless $k = 0$, in which case it is equal to 1. Therefore, the only surviving part is the one with $i = t - l + 1$:

$$\xi^{-\sum_{j=n-t+l-1}^{n-1} j} = \xi^{\binom{n}{2} - \binom{n-1-t+l}{2}} = \xi^{-c(t,l)}.$$

The proofs of the remaining two cases, when l is odd, are very similar. The only difference in the case $2n - 2 \leq t - l$ is that first the preimages with zero or one extra movable beads at R_{n-1} have to be counted, then the preimages with two or three extra movable beads, etc. As in the case $t \leq n - 1$, the only nonzero term is the one where in the beginning all the movable beads have been shifted to the R_{n-1}-st ruler, and in this case the powers of ξ^{-1} sum up to $\binom{n}{2} = c(t,l)$. □

COROLLARY 9.11. *Let $Y \in \mathcal{Z}_\Delta^0$ be a distinguished 0-generated Young wall described by the data $\{(t_i, l_i)_i\}$. Then*

$$\sum_{Y' \in p_*^{-1}(Y)} q^{\underline{\mathrm{wt}}(Y')} \Big|_{q_1 = \cdots = q_n = \xi,\, q_0 = \xi^2 q} = q^{\underline{\mathrm{wt}}(Y)} \Big|_{q_1 = \cdots = q_n = \xi,\, q_0 = \xi^{-(2n-3)} q} \cdot \xi^{-\sum_i c(t_i, l_i)}$$

$$= q^{\mathrm{wt}_0(Y)} \xi^{\sum_{j \neq 0} (\mathrm{wt}_j(Y) - \dim \rho_j \cdot \mathrm{wt}_0(Y)) - \sum_i c(t_i, l_i)}$$

LEMMA 9.12. *The core of a 0-generated Young wall is a 0-generated Young wall.*

PROOF. With each reduction step (B1)-(B4) we always remove a bar. In the original Young wall, the salient blocks were only label 0 half blocks and salient block-pairs of label $n - 1/n$.

A similar reasoning as in the type A case shows that no salient block of label $2, \ldots, n - 2$ can appear after we perform the step (B1) until possible. The same is true with (B2), since if we can perform it on a pair of beads in a row, then we can always perform it on the beads between them. More precisely, it can be seen that even label 1 salient blocks cannot appear during these two steps because the parity conditions in (D1) and (D2) is always maintained by the reduction steps.

The parity conditions in (D1) and (D2) are maintained by the step (B3) as well. If we perform (B4) on a pair of beads, then we can perform it on this pair as long as they disappear from the abacus. Hence, when performing (B4) until possible we also get back the good parities. After the reduction is completed there cannot be any bead on the rulers $R_1, \ldots, R_{n-2}$, and all the beads on the rulers $R_n, \ldots, R_{2n-3}$ are right-adjusted. Therefore the conditions (D4) and (D5) are also satisfied.

If the total number of beads was initially even, then since no label 1 salient block can appear, the total number of beads in the end must be even as well. So the final Young wall will satisfy (D3). If in the total number of beads in the initial abacus configuration is odd, then in the first row $t_1 - l_1 = 2n - 4$. Hence, the beads on ruler R_{2n-2} in the first row are necessarily white and one of them can be taken away from the abacus with the step (B3), together with the beads on the other rulers using (B2). This is an odd number of beads removed from the abacus. After this the total number of beads is even, so we reduced to the earlier case. □

PROOF OF THEOREM 9.8. In light of Corollary 9.11, it remains to show that

$$\xi^{\sum_{j\neq 0}(\mathrm{wt}_j(Y)-\dim\rho_j\cdot\mathrm{wt}_0(Y))-\sum_i c(t_i,l_i)} = 1.$$

We follow in the line of the proof the A_n case.

Step 1: Reduction to 0-generated cores. According to Lemma 9.12 the core of a 0-generated Young wall is a 0-generated core. It is immediate from the definition of $c(t,l)$ that the steps (B1), (B2) and (B4) leave the sum $\sum_i c(t_i, l_i)$ unchanged, while (B3) is a bit more complicated. Indeed,

- (B1) removes one movable bead from a row, and adds one to another on the same ruler;
- (B2) removes two movable beads, but these two contribute with opposite signs into $c(t, l)$;
- (B4) either moves non-movable beads from R_{2n-2} onto R_{n-1}, or beads from R_{n-1} into non-movable beads on R_{2n-2}.

Moving beads on R_{n-1} according to (B3) does not affect the numbers $c(t, l)$. If (B3) moves a bead on the ruler R_{2n-2} between rows having l of different parity, then the sum of the movable beads at the $(2n-2)$-nd positions of the two rows is constant, so $\sum_i c(t_i, l_i)$ does not change after such a step. If (B3) were able to move a bead on R_{2n-2} between rows both having odd l's, then this can happen only if they have the same color and if there is no bead on R_{n-1} between them. This means that there must be an even number of beads between them and they should have same kind of top block, or odd number of beads and different kind of top blocks. Both cases are forbidden in 0-generated Young walls due to Lemma 9.6. For the same reasons (B3) cannot move beads on R_{2n-2} between rows both having even l's.

It follows from these considerations that for all $Y \in \mathcal{Z}_\Delta^0$

$$\xi^{\sum_{j\neq 0}(\mathrm{wt}_j(Y)-\dim\rho_j\cdot\mathrm{wt}_0(Y))-\sum_i c(t_i,l_i)} = \xi^{\sum_{j\neq 0}(\mathrm{wt}_j(\mathrm{core}(Y))-\dim\rho_j\cdot\mathrm{wt}_0(\mathrm{core}(Y)))-\sum_i c(t_i,l_i)}.$$
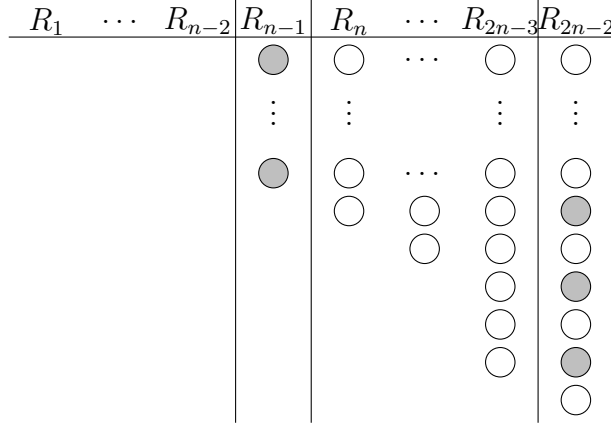
Step 2: Reduction to distinguished 0-generated cores.

As described above, for any 0-generated core Y there is a decomposition as $\lambda(Y) = \mu(Y) \cup \nu(Y)$, where $\mu(Y) \in \mathcal{C}^1$, $\nu(Y) \in \mathcal{C}^2$, and we have $\nu(Y) = \nu^{(0)}(Y) + \nu^{(1)}(Y)$, where $\nu^{(0)}(Y)$ and $\nu^{(1)}(Y)$ are two-cores, and parts in $\nu^{(1)}(Y)$ have colors given by their parity. Since Y is 0-generated, the largest part of $\nu(Y)$ has to be even, otherwise there is no bead at the rightmost position in the last row of the abacus. Therefore, the abacus represenation of Y can be described as follows.

(1) There is no bead on the rulers R_k for $1 \leq k \leq n-2$;
(2) On the ruler R_{n-1} all the beads are of the same color, the beads are at the first, let's say, m positions, exactly one bead at each.
(3) The positions in the first m rows of the rulers R_k for $n \leq k \leq 2n-3$ are all filled up with beads, the other beads are pushed to the right as much as possible.

(4) There is at least m beads on the ruler R_{2n-2}, one at each of the first positions and there is no space between them. The first m of these are all white. The total number of the rest is even, and half of them is white, half of them is black.

The abacus of a typical 0-generated core looks like this:

It is not true that the core of a distinguished 0-generated Young wall is always distinguished. But we can reduce each non-distinguished core further using the reduction map red and then taking the core of the result using the steps (B1)-(B4) again. The result of this is a distinguished 0-generated core. The first step corresponds to shifting the beads on R_{n-1} one step left. This decreases $\sum_i c(t_i, l_i)$ by m, and decreases $\sum_{j\neq 0}(\mathrm{wt}_j(Y) - \dim \rho_j \cdot \mathrm{wt}_0(Y))$ also by m. The second step does not change these numbers by the considerations of Step 1. So we are done once we know the statement for distinguished 0-generated cores.

We remark here that during the first step the color of every second bead on the ruler R_{2n-2} changes. In the end the color of the beads on this ruler will alternate.

Step 3: The case of distinguished 0-generated cores.

Let $(z_1, \ldots, z_n)$ be the integer tuple assigned to Y as in 9.3. Then $z_{n-1} = z_n$ since there is no bead on R_{n-1} and the color of the beads on R_{2n-2} alternates. In particular, $z_{n-1} + z_n$ is an even number.

We claim that

$$\sum_{1=1}^{n} m_n = \sum_i c(t_i, l_i).$$

Indeed, by Theorem 9.3 (3), $\sum_{1=1}^{n} m_n = -\sum_{i=1}^{n-2}(n-1-i)z_i - (n-1)c(z_{n-1} + z_n) - (n-1)b$. The number of movable beads on R_{2n-2-i} is $-z_i$ if $1 \leq i \leq n-1$, and it is $-c(z_{n-1} + z_n) - b$ if $i = 0$. This proves the claim.

By Theorem 9.3 (1), for a core

$$\underline{q}^{\underline{\mathrm{wt}}(Y)} = q_1^{m_1} \cdot \cdots \cdot q_n^{m_n}(q^{1/2})^{\overline{m}^\top \cdot C \cdot \overline{m}}.$$

As in the type A case, on the right hand side of this expression q_0 only appears in the $q^{1/2}$-term. Hence

$$\frac{1}{2}\overline{m}^\top \cdot C \cdot \overline{m} = \mathrm{wt}_0(Y),$$

and

$$\left.(q^{1/2})^{\overline{m}^\top \cdot C \cdot \overline{m}}\right|_{q_1=\cdots=q_n=\xi,\, q_0=\xi^2 q} = q^{\mathrm{wt}_0(Y)}.$$

On the other hand,

$$\left.q^{\underline{\mathrm{wt}}(Y)}\right|_{q_1=\cdots=q_n=\xi,\, q_0=\xi^2 q} = q^{\mathrm{wt}_0(Y)}\xi^{\sum_{j\neq 0}(\mathrm{wt}_j(Y)-\dim\rho_j\cdot\mathrm{wt}_0(Y))}.$$

Therefore

$$\xi^{\sum_{j\neq 0}(\mathrm{wt}_j(Y)-\dim\rho_j\cdot\mathrm{wt}_0(Y))} = \xi^{\sum_{i=1}^n m_n} = \xi^{\sum_i c(t_i,l_i)},$$

and so indeed

$$\xi^{\sum_{j\neq 0}(\mathrm{wt}_j(Y)-\dim\rho_j\cdot\mathrm{wt}_0(Y))-\sum_i c(t_i,l_i)} = 1.$$

$\square$

Bibliography

1. A. B. Altman and S. L. Kleiman, *Joins of schemes, linear projections*, Compos. Math. **31** (1975), no. 3, 309–343.

2. G. E. Andrews, *Generalized Forbenius Partitions*, Mem. Amer. Math. Soc., vol. 301, Amer. Math. Soc., Providence, 1984.

3. ———, *The theory of partitions*, Encyclopedia of Mathematics and its Applications, vol. 2, Cambridge University Press, 1998.

4. A. Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (1973), 480–497.

5. J. Briançon, *Description de* $\mathrm{Hilb}^n\mathbb{C}[x,y]$, Invent. Math. **41** (1977), no. 1, 45–89.

6. M. Brion, *Invariant Hilbert schemes*, Handbook of moduli. Vol. I, Adv. Lect. Math., vol. 24, Int. Press, Somerville, MA, 2013, pp. 64–117.

7. H. Cartan, *Quotient d'un espace analytique par un groupe d'automorphismes*, Algebraic geometry and topology: a symposium in honor of S. Lefschetz, Princeton University Press, Princeton, NJ, 1957, pp. 90–102.

8. J. Cheah, *On the cohomology of Hilbert schemes of points*, J. Algebraic Geom. **5** (1996), no. 3, 479–512.

9. H.-J. Chung, T. Dimofte, S. Gukov, and P. Sułkowski, *3d-3d Correspondence Revisited*, J. High Energy Phys. **2016** (2016), no. 4, Art–No.

10. A. N. de Celis, *Dihedral groups and G-Hilbert schemes*, Ph.D. thesis, University of Warwick, 2008.

11. R. Dijkgraaf, L. Hollands, P. Sułkowski, and C. Vafa, *Supersymmetric gauge theories, intersecting branes and free fermions*, J. High Energy Phys. **2008** (2008), no. 02, 106.

12. R. Dijkgraaf and P. Sułkowski, *Instantons on ALE spaces and orbifold partitions*, J. High Energy Phys. **3** (2008), 013–013.

13. D. Eisenbud and J. Harris, *3264 and All That: A Second Course in Algebraic Geometry*, Cambridge University Press, 2016.

14. G. Ellingsrud and S. A. Strømme, *On the homology of the Hilbert scheme of points in the plane*, Inven. Math. **87** (1987), no. 2, 343–352.

15. P. Etingof, *Symplectic reflection algebras and affine Lie algebras*, Mosc. Math. J **12** (2012), no. 3, 543–565.

16. J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math. **90** (1968), no. 2, 511–521.

17. I. B. Frenkel and V. G. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62** (1980), no. 1, 23–66.

18. Sh. Fujii and S. Minabe, *A combinatorial study on quiver varieties*, `http://arxiv.org/abs/math/0510455`, 2005.

19. F. Garvan, D. Kim, and D. Stanton, *Cranks and t-cores*, Invent. Math. **101** (1990), no. 1, 1–17.

20. I. G. Gordon, *Quiver varieties, category $\mathcal{O}$ for rational Cherednik algebras, and Hecke algebras*, Int. Math. Res. Pap. (2008), no. 3, rpn006.

21. L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. **286** (1990), no. 1, 193–207.

22. _____, *Invariants of moduli spaces and modular forms*, Rend. Istit. Mat. Univ. Trieste **41** (2009), 55–76.

23. I. Grojnowski, *Instantons and affine algebras I: The Hilbert scheme and vertex operators*, Math. Res. Lett. **3** (1996), no. 2, 275–292.

24. A. Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébrique IV: Les schémas de Hilbert*, Séminaire Bourbaki **6** (1960), 249–276.

25. S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández, *On generating series of classes of equivariant Hilbert schemes of fat points*, Mosc. Math. J. **10** (2010), no. 3, 593–602.

26. Á. Gyenge, *Enumeration of diagonally colored Young diagrams*, `http://arxiv.org/abs/1510.02677`, 2015.

27. _____, *Hilbert scheme of points on cyclic quotient singularities of type $(p, 1)$*, `http://arxiv.org/abs/1603.02114`, 2016.

28. Á. Gyenge, A. Némethi, and B. Szendrői, *Euler characteristics of Hilbert schemes of points on simple surface singularities*, `http://arxiv.org/1512.06848`, 2015.

29. _____, *Euler characteristics of Hilbert schemes of points on surfaces with simple singularities*, `http://arxiv.org/1512.06844`, 2015.

30. R. Hartshorne, *Connectedness of the Hilbert scheme*, Publ. Math. Inst. Hautes Etudes Sci. **29** (1966), 5–48.

31. _____, *Algebraic geometry*, Graduate Text in Mathematics, vol. 52, Springer, 1977.

32. J. Hong and S.-J. Kang, *Introduction to quantum groups and crystal bases*, Graduate Studies in Mathematics, vol. 42, American Mathematical Society, 2002.

33. D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Cambridge University Press, 2010.

34. A. A. Iarrobino, *Punctual Hilbert schemes*, Mem. Amer. Math. Soc, vol. 188, American Mathematical Society, 1977.

35. Y. Ito and I. Nakamura, *Hilbert schemes and simple singularities*, London Mathematical Society Lecture Note Series (1999), 151–234.

36. V. G. Kac, *Infinite-dimensional Lie algebras*, Cambridge University Press, 1994.

37. S.-J. Kang, *Crystal bases for quantum affine algebras and combinatorics of Young walls*, Proc. London Math. Soc. **86** (2003), no. 1, 29–69.

38. S.-J. Kang and J.-H. Kwon, *Quantum affine algebras, combinatorics of Young walls, and global bases*, Electron. Res. Announc. Amer. Math. Soc. **8** (2002), 35–46.

39. _____ , *Crystal bases of the Fock space representations and string functions*, J. Algebra **280** (2004), no. 1, 313–349.

40. M. Kapranov and E. Vasserot, *Kleinian singularities, derived categories and Hall algebras*, Math. Ann. **316** (2000), no. 3, 565–576.

41. J.-H. Kwon, *Affine crystal graphs and two-colored partitions*, Lett. Math. Phys. **75** (2006), no. 2, 171–186.

42. S. Lang, *Algebra, Revised third edition*, Graduate Texts in Mathematics, vol. 211, Springer, 2002.

43. A. Lascoux, B. Leclerc, and J.-Y. Thibon, *Hecke algebras at roots of unity and crystal bases of quantum affine algebras*, Comm, Math. Phys. **181** (1996), no. 1, 205–263.

44. B. Leclerc and H. Miyachi, *Some closed formulas for canonical bases of Fock spaces*, Represent. Theory **6** (2002), no. 2002, 290–312.

45. O. Lorscheid and Th. Weist, *Quiver Grassmannians of extended Dynkin type D - Part 1: Schubert systems and decompositions into affine spaces*, `http://arxiv.org/abs/1507.00392`, 2015.

46. _____ , *Quiver Grassmannians of extended Dynkin type D - Part 2: Schubert decompositions and F-polynomials*, `http://arxiv.org/abs/1507.00395`, 2015.

47. I. G. Macdonald, *The Poincaré polynomial of a symmetric product*, Math. Proc. Cambridge Philos. Soc. **58** (1962), no. 4, 563–568.

48. D. Maulik, *Stable pairs and the HOMFLY polynomial*, Invent. Math. (2012), 1–45.

49. K. C. Misra and T. Miwa, *Crystal base for the basic representation of $U_q(\widehat{\mathfrak{sl}}(n))$*, Comm. Math. Phys. **134** (1990), no. 1, 79–88.

50. K. Nagao, *Quiver varieties and Frenkel–Kac construction*, J. Algebra **321** (2009), no. 12, 3764–3789.

51. H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416.

52. _____ , *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. of Math. (2) **145** (1997), no. 2, 379–388.

53. _____ , *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, vol. 18, Amer. Math. Soc., Providence, 1999.

54. _____ , *Quiver varieties and finite dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), no. 1, 145–238.

55. _____ , *Geometric construction of representations of affine algebras*, Proceedings of the International Congress of Mathematicians (Beijing, 2002), vol. 1, IMU, Higher Ed, 2002, pp. 423–438.

56. H. Nakajima and K. Yoshioka, *Instanton counting on blowup. I. 4-dimensional pure gauge theory*, Invent. Math. **162** (2005), no. 2, 313–355.

57. A. Oblomkov and V. Shende, *The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link*, Duke Math. J. **161** (2012), no. 7, 1277–1303.

58. A. M. Odlyzko and H. S. Wilf, *The editor's corner: n coins in a fountain*, Amer. Math. Monthly **95** (1988), no. 9, 840–843.

59. S. V. Sam and P. Tingley, *Combinatorial realizations of crystals via torus actions on quiver varieties*, J. Algebraic Combin. **39** (2014), no. 2, 271–300.

60. A. Savage and P. Tingley, *Quiver grassmannians, quiver varieties and the preprojective algebra*, Pacific J. Math. **251** (2011), no. 2, 393–429.

61. P. Tingley, *Notes on Fock Space*, `http://webpages.math.luc.edu/~ptingley/`, 2011.

62. Y. Toda, *S-Duality for surfaces with A_n-type singularities*, Math. Ann. **363** (2015), 679–699.

63. C. Vafa and E. Witten, *A strong coupling test of S-duality*, Nucl. Phys. B **431** (1994), no. 1, 3–77.

64. B. Young and J. Bryan, *Generating functions for colored 3D Young diagrams and the Donaldson-Thomas invariants of orbifolds*, Duke Math. J. **152** (2010), no. 1, 115–153.

Summary

The punctual Hilbert scheme parameterizing the zero-dimensional subschemes of a quasi-projective variety contains a large amount of information about the geometry and topology of the base variety. The Hilbert schemes of points on smooth curves and surfaces have been investigated for a long time. In the recent years a new direction has emerged, which also allows singularities on the base variety. The aim of this thesis is to describe the Euler characteristics of the Hilbert schemes parameterizing the zero-dimensional subschemes of some basic classes of surface singularities.

The well-known simple singularities are the simplest type of normal surface singularities, and it is known that they have an orbifold structure. There are at least two natural version of the punctual Hilbert scheme in the case of quotient singularities.

We study the geometry and topology of Hilbert schemes of points on the orbifold surface $[C^2/G]$, respectively the singular quotient surface C^2/G, where G is a finite subgroup of $SL(2, C)$ of type A or D. We give a decomposition of the (equivariant) Hilbert scheme of the orbifold into affine space strata indexed by a certain combinatorial set, the set of Young walls. The generating series of Euler characteristics of Hilbert schemes of points of the singular surface of type A or D is computed in terms of an explicit formula involving a specialized character of the basic representation of the corresponding affine Lie algebra; we conjecture that the same result holds also in type E. Our results are consistent with known results for type A, and are new for type D. The crystal basis theory of the fundamental representation of the affine Lie algebra corresponding to the surface singularity (via the McKay correspondence) plays an important role in our approach. The result gives a generalization of Göttsche's formula and has interesting modular properties related to the S-duality conjecture.

The moduli space of torsion free sheaves on surfaces are higher rank analogs of the Hilbert schemes. In type A our results reveal their Euler characteristic generating function as well. Another very interesting class of normal surface singularities is the so-called cyclic quotient singularities of type $(p, 1)$. As an outlook we also obtain some results about the associated generating functions.

Összefoglaló

Egy kváziprojektív varietás nulla dimenziós részsémáit paraméterező pontozott Hilbert séma nagy mennyiségű információt tartalmaz az alapvarietás geometriájáról és topológiájáról. Sima görbék és felületek pontjainak Hilbert sémáit régóta vizsgálják. Az utóbbi években egy új irányzat is megjelent, miszerint az alapvarietáson szingularitások is megengedettek. A disszertáció célja, hogy leírja néhány alap felületszingularitás osztály eseten a nulla dimenziós részsémákat paraméterező Hilbert-sémák Euler-karakterisztikáit.

Az egyszerű szingularitások a legegyszerűbbek a normális felületszingularitások között, és ismert hogy orbifold struktúrával is rendelkeznek. Hányadosszingularitások esetén legalább két természetes verziója létezik a pontozott Hilbert-sémának.

Megvizsgáljuk a $[C^2/G]$ orbifold felület, és a C^2/G hányadosfelület pontjainak Hilbert-sémáját, ahol G az $SL(2, C)$-nek A vagy D típusú részcsoportja. Az ekvivariáns Hilbert-sémáknak megadjuk egy felbontását affin terekre, amiket Young falaknak egy meghatározott kombinatorikusan leírható halmaza indexel. Explicit alakban kiszámoljuk az A és D típusú szinguláris felületek Hilbert-sémáinak Euler-karakterisztikáinak generátorfüggvényét is, ami egy specializációja a megfelelő affin Lie-algebra alapreprezentációjának karakterének. Az E típus esetét sejtésként fogalmazzuk meg. Az eredményeink konzisztensek a korábbiakkal az A típusra, és újak a D típusra. A megközelítésünkben fontos szerepet játszik az felületszingularitáshoz a McKay-korrespondencián keresztül tartozó affin Lie-algebra alapreprezentációjának kristálybázis elmélete. Az eredmények általánosítják Göttsche formuláját és az S-dualitás sejtéshez kapcsolódóan érdekes modularitási tulajdonságokkal is rendelkeznek.

A torziómentes kévék modulustere a pontozott Hilbert séma egy magasabb dimenziós analógiája. Az A típus esetében az itt ismertetett módszer megadja ezen modulusterek Euler-karakterisztikáinak generátorfüggvényét is. A normális felületszingularitások egy másik, nagyon érdekes osztálya a $(p, 1)$-típusú ciklikus szingularitásoké. Kitekintésként az ezekhez asszociált generátorfüggvényekről is bemutatunk néhány eredményt.