Commutative algebra and algebraic geometry
Sheet 1 — Due 23/09

Practice problems (not to hand in)

1. Let x be a nilpotent element of a ring A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.
2. Show that if A is a ring, the set of zero-divisors in A is a union of prime ideals of A.
3. Show that $\sqrt{I}$ is the intersection of the prime ideals containing I. In particular, it's an ideal containing I.
4. Let X be an affine variety. Show that the coordinate ring $A(X)$ is a field if and only if X is a single point.

HW problems to hand in

1. Let A be a ring and let $A[x]$ be the ring of polynomials over it. Let $f = a_0 + a_1 x + \cdots + a_n x^n \in A[x]$. Show that
   (a) f is a unit in $A[x] \iff a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent. *Hint:* If $b_0 + b_1 x + \cdots + b_m x^m$ is the inverse of f, prove by induction on r that $a_n^{r+1} b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Practice Exercise 1.
   (b) f nilpotent $\iff a_0, \ldots, a_n$ are nilpotent
   (c) f zero-divisor $\iff$ there exists $a \neq 0$ in A such that $af = 0$.
   Hint: Choose a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. Then $a_n b_m = 0$, hence $a_n g = 0$, because $a_n g$ annihilates f and has degree $< m$. Now show by induction that $a_{n-r} g = 0$, $0 \leq r \leq n$.
2. Let A be a ring. Show that in the ring $A[x]$ the Jacobson radical is equal to the nilradical.
3. Let $A = F_3[x, y]/(xy, x^2)$. Give as explicit a list as possible of all the prime ideals in R. Which ones are maximal? What are the Jacobson radical and nilradical of A?
4. Show that the equation of ideals
$$(x^3 - x^2, x^2 y - x^2, xy - y, y^2 - y) = (x^2, y) \cap (x - 1, y - 1)$$
   holds in the polynomial ring $\mathbb{C}[x, y]$. Is this a radical ideal? What is its zero locus in $\mathbb{A}^2_{\mathbb{C}}$?
5. Let $X \subset \mathbb{A}^3$ be the union of the three coordinate axes. Compute generators for the ideal $I(X)$. Show that $I(X)$ cannot be generated by fewer than three elements. *Hint:* any element $p(x, y, z) \in \mathbb{C}[x, y, z]$ is of the form $p(x, y, z) = \sum_{i,j,k} a_{ijk} x^i y^j z^k$.