Commutative algebra and algebraic geometry
Sheet 2 — Due 07/10

Practice problems (not to hand in)

1. Let $X \subset \mathbb{A}^n$ be an arbitrary subset. Prove that $V(I(X)) = \overline{X}$, the closure of X.
2. Let k be an infinite field, $C := \{(t, t^3, t^4)\} \subset \mathbb{A}^3_k$. Show that C is an algebraic variety, and determine its vanishing ideal.
3. Decide if the following statements are true or false.
   (a) Let X be a topological space, $Y, Z \subseteq X$ irreducible subspaces. Then $Y \cap Z$ is irreducible as well.
   (b) The affine variety corresponding to $\mathbb{C}[x, y]/(xy)$ is irreducible.
   (c) Every open subset of A^2 in the Zariski topology is dense.
   (d) The affine variety corresponding to $\mathbb{C}[x, y](xy - 1)$ is irreducible.

HW problems to hand in

1. Let $A = \mathbb{C}[x, y]/(x - y^2)$. Find a minimal primary decomposition of the ideal I in A generated by [the image of] $x - 1$, resp., $(x - 1)^2$.
2. Show that ideal quotients correspond to differences of varieties in the following sense: if X is an affine variety and
   (a) Y_1 and Y_2 are subvarieties of X then $I(\overline{Y_1 \setminus Y_2}) = (I(Y_1) : I(Y_2))$ in $A(X)$
   (b) J_1 and J_2 are radical ideals in $A(X)$ then $\overline{V(J_1) \setminus V(J_2)} = V(J_1 : J_2)$
3. Let X be a topological space. Let $X_1, \ldots, X_r$ be any subset of X with $X = X_1 \cup \cdots \cup X_r$. If all $X_1, \ldots, X_r$ are Noetherian, prove that X is Noetherian as well.
4. (a) List the open and closed subsets of $\mathbb{A}^1$ in the Zariski topology.
   (b) Show that the only irreducible closed algebraic sets of $\mathbb{A}^2$ are $\mathbb{A}^2$, $\emptyset$, $V(x - a, y - b)$, and $V(f(x, y))$ where $f(x, y)$ is an irreducible polynomial.
   (c) Deduce that the dimension of $\mathbb{A}^2$ is 2.
   (d) Describe carefully all the Zariski closed subsets of $\mathbb{A}^2$.
   (e) Show that the Zariski topology on $\mathbb{A}^2$ is *not* the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$.