Commutative algebra and algebraic geometry

Sheet 3 — Due 28/10

Practice problems (not to hand in)

1. Let A be a ring, let S and T be two multiplicatively closed subsets of A, and let U be the image of T in $S^{-1}A$. Show that the rings $(ST)^{-1}A$ and $U^{-1}(S^{-1}A)$ are isomorphic.
2. Let $\mathcal{F}$ be a sheaf on a topological space X, and let $a \in X$. Show that the stalk $\mathcal{F}_a$ is a local object in the following sense: If $U \subset X$ is an open neighborhood of a then $\mathcal{F}_a$ is isomorphic to the stalk of $\mathcal{F}|_U$ at a on the topological space U.
3. Let $f : X \to Y$ be a morphism of affine varieties and $f^* : A(Y) \to A(X)$ the corresponding homomorphism of the coordinate rings. Are the following statements true or false?
   (a) $f$is surjective if and only if f^* is injective.
   (b) f is injective if and only if f^* is surjective.
   (c) If $f : \mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism then f is affine linear, i. e. of the form $f(x) = ax + b$ for some $a, b \in k$.
   (d) If $f : \mathbb{A}^2 \to \mathbb{A}^2$ is an isomorphism then f is affine linear, i. e. it is of the form $f(x) = Ax+b$ for some $A \in Mat(2 \times 2, k)$ and $b \in k^2$.
4. Show that the space $\mathbb{P}^1$ is a variety (we have seen in the class that it is a prevariety).

HW problems to hand in

1. Let R be a local ring with maximal ideal P. Show that the localization map $R \to R_P$ is an isomorphism.
2. Let $R = \mathbb{Z}[x]/(x^2 - 1)$. Show that $J = (x - 1)$ is a prime ideal in R and show that R_J is isomorphic to $\mathbb{Q}$. What is the kernel of the map $R \to R_J$?
3. Let $\phi, \psi \in \mathcal{F}(U)$ be two sections of a sheaf $\mathcal{F}$ on an open subset U of a topological space X. Show:
   (a) If ϕ, ψ agree in all stalks, i.e. $\overline{(U, \phi)} = \overline{(U, \psi)} \in \mathcal{F}_a$ for all $a \in U$, then $\phi = \psi$.
   (b) If $\mathcal{F} = \mathcal{O}_X$ is the sheaf of regular functions on an irreducible affine variety X then we can already conclude that $\phi = \psi$ if we only know that they agree in one stalk $\mathcal{F}_a$ for $a \in U$.
   (c) For a general sheaf $\mathcal{F}$ on a topological space X the statement of (b) is false.
4. An irreducible quadric curve in $\mathbb{A}^2$ is also called an *affine conic*. Show that every affine conic over a field of characteristic not equal to 2 is isomorphic to exactly one of the varieties $X_1 = V(x_2 - x_1^2)$ and $X_2 = V(x_1 x_2 - 1)$, with an isomorphism given by a linear coordinate transformation followed by a translation.
5. Let X and Y be prevarieties. Show that if X and Y are varieties then so is $X \times Y$.