Commutative algebra and algebraic geometry

Sheet 5 — Due 02/12

Practice problems (not to hand in)

1. Let $\widetilde{\mathbb{A}^3}$ be the blow-up of $\mathbb{A}^3$ at the line $V(x_1, x_2) \simeq \mathbb{A}^1$. Show that its exceptional set is isomorphic to $\mathbb{A}^1 \times \mathbb{P}^1$. When do the strict transform of two lines in $\mathbb{A}^3$ through $V(x_1, x_2)$ intersect in the blow-up?

2. (a) Show that if $\mathrm{char}(k)$ does not divide d, then the hypersurface $V(x_0^d + \ldots + x_n^d) \subset \mathbb{P}^n$ is nonsingular.

   (b) Under the same assumption on k, find the singular locus of the hypersurface $V(x_0^d + \ldots + x_{n-1}^d) \subset \mathbb{P}^n$.

HW problems to hand in

1. The *incidence correspondence* is the set of pairs consisting of a $L \subset \mathbb{P}^{n-1}$ along with a point $a \in L$ contained in the linear space:

$$\{(L, a) \in G(k, n) \times \mathbb{P}^{n-1} : a \in L\}$$

   Show that it is a projective variety.

2. Let $X \subset \mathbb{P}^n$ be a quadric, i.e. an irreducible variety which is the zero locus of an irreducible homogeneous polynomial of degree 2. Show that X is birational to, but not in general isomorphic to, $\mathbb{P}^{n-1}$.

3. Let $f : X \to Y$ be a morphism of varieties, and let $a \in X$. Show that f induces a linear map $T_a X \to T_{f(a)} Y$ between tangent spaces.

4. Consider the surface $S_2 = V(-xy + z^2) \subset \mathbb{A}^3$. Let $\widetilde{S_2}$ be its blow-up in the origin.

   (a) Verify that S_2 has a singular point at the origin.

   (b) Show that $\widetilde{S_2}$ is smooth, and hence $\pi : \widetilde{S_2} \to S_2$ is a resolution of singularities.

   (c) What is the exceptional set of π?