

Commutative algebra and algebraic geometry

Sheet 2 — Due 20/10

Practice problems (not to hand in)

1. Let $X \subset \mathbb{A}^n$ be an arbitrary subset. Prove that $V(I(X)) = \overline{X}$, the closure of X .
 2. Let k be an infinite field, $C := \{(t, t^3, t^4)\} \subset \mathbb{A}_k^3$. Show that C is an algebraic variety, and determine its vanishing ideal.
 3. Decide if the following statements are true or false.
 - (a) Let X be a topological space, $Y, Z \subseteq X$ irreducible subspaces. Then $Y \cap Z$ is irreducible as well.
 - (b) The affine variety corresponding to $\mathbb{C}[x, y]/(xy)$ is irreducible.
 - (c) Every open subset of $\mathbb{A}^2$ in the Zariski topology is dense.
 - (d) The affine variety corresponding to $\mathbb{C}[x, y](xy - 1)$ is irreducible.
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HW problems to hand in

1. Let $A = \mathbb{C}[x, y]/(x - y^2)$. Find a minimal primary decomposition of the ideal I in A generated by [the image of] $x - 1$, resp., $(x - 1)^2$.
2. Show that ideal quotients correspond to differences of varieties in the following sense: if X is an affine variety and
 - (a) Y_1 and Y_2 are subvarieties of X then $I(\overline{Y_1 \setminus Y_2}) = (I(Y_1) : I(Y_2))$ in $A(X)$
 - (b) J_1 and J_2 are radical ideals in $A(X)$ then $\overline{V(J_1) \setminus V(J_2)} = V(J_1 : J_2)$
3. Let X be a topological space. Let $X_1, \dots, X_r$ be any subset of X with $X = X_1 \cup \dots \cup X_r$. If all $X_1, \dots, X_r$ are Noetherian, prove that X is Noetherian as well.
4.
 - (a) List the open and closed subsets of $\mathbb{A}^1$ in the Zariski topology.
 - (b) Show that the only irreducible closed algebraic sets of $\mathbb{A}^2$ are $\mathbb{A}^2$, $\emptyset$, $V(x - a, y - b)$, and $V(f(x, y))$ where $f(x, y)$ is an irreducible polynomial.
 - (c) Deduce that the dimension of $\mathbb{A}^2$ is 2.
 - (d) Describe carefully all the Zariski closed subsets of $\mathbb{A}^2$.
 - (e) Show that the Zariski topology on $\mathbb{A}^2$ is *not* the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$.