Commutative algebra and algebraic geometry
Sheet 4 — Due 26/11

Practice problems (not to hand in)

1. Let $a \in \mathbb{P}^2$ be a point. Show that the one-point set $\{a\}$ is a projective variety, and compute explicit generators for the ideal $I_p(\{a\})$ of $k[x_0, \ldots, x_n]$.
2. Sketch the set of real points of the complex affine curve $X = V(x_1^3 - x_1x_2^2 + 1) \subset \mathbb{A}^2_{\mathbb{C}}$ and compute the points at infinity of its projective closure $\overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$.
3. Show that every morphism $f : \mathbb{P}^n \to \mathbb{P}^m$ must be of the form
$$f : \mathbb{P}^n \to \mathbb{P}^m, x \mapsto (f_0(x) : \cdots : f_m(x))$$
with $f_0, \ldots, f_m \in k[x_0, \ldots, x_n]$ homogeneous polynomials of the same degree such that
$$V_p(f_0, \ldots, f_m) = \emptyset.$$

HW problems to hand in

1. (a) Prove that a graded ring R is an integral domain if and only if for all homogeneous elements $f, g \in R$ with $fg = 0$ we have $f = 0$ or $g = 0$.
   (b) Show that a projective variety is irreducible if and only if its homogeneous coordinate ring $S(X)$ is an integral domain.
2. Let $X \subset \mathbb{P}^2$ be a curve given as the zero locus of a homogeneous polynomial of degree 3, that is, a cubic. Let $U \subset X \times X$ be the set of all $(a, b) \in X \times X$ such that $a \neq b$ and the unique line through the two points a and b meets X in exactly three distinct points. Of course, two of these points must be a and b; denote the third one by $f(a, b) \in X$. Show that $U \subset X \times X$ is open and that $f : U \to X$ is a morphism.
3. Let $X \subset \mathbb{P}^n$ be a connected projective variety and let $V \subset \mathbb{A}^m$ be an affine variety. Let $\phi : X \to V$ be a morphism. Prove that ϕ is constant, i.e. there exists a $v \in V$ such that $\phi(x) = v$ for all $x \in X$.
4. The projective variety $C = V(F_0, F_1, F_2) \subset \mathbb{P}^3$, where
$$\begin{aligned} F_0(z_0, z_1, z_2, z_3) &= z_0z_2 - z_1^2 \\ F_1(z_0, z_1, z_2, z_3) &= z_0z_3 - z_1z_2 \\ F_2(z_0, z_1, z_2, z_3) &= z_1z_3 - z_2^2, \end{aligned}$$
is known as the *twisted cubic.*
   (a) Show that C is equal to the image of the map
$$\begin{aligned} \nu : \mathbb{P}^1 &\to \mathbb{P}^3 \\ \nu : [x_0 : x_1] &\mapsto [x_0^3 : x_0^2x_1 : x_0x_1^2 : x_1^3]. \end{aligned}$$
   (b) Restrict to the affine patch $U_0 \subset \mathbb{P}^3$ given by setting $z_0 = 1$. Show that $C \cap U_0$ is equal to $V(f_0, f_1) \subset \mathbb{A}^3$, where $f_i(z_1, z_2, z_3) := F_i(1, z_1, z_2, z_3)$ for $i = 1, 2$.
   (c) For $i = 0, 1, 2$ we write Q_i for the quadric hypersurface $V(F_i) \subset \mathbb{P}^3$. Show that, for $i \neq j$, the hypersurfaces Q_i and Q_j intersect in the union of C and a line L. Therefore no two of them alone may be used to define C. Deduce that the homogenizations of the generators of an affine ideal do not necessarily generate the homogeneous ideal of the projective closure, showing that indeed we need to homogenise *all* elements of the affine ideal.