Algebraic topology and homological algebra
Sheet 2 — Due 25/03

Practice problems (not to hand in)

1. Compute $H_i(S^n \setminus k+1 \text{ points })$ and $H_i(\mathbb{R}^n \setminus k \text{ points })$.
2. Show that the relative homology group $H_1(\mathbb{R}, \mathbb{Q})$ of the pair $\mathbb{Q} \subset \mathbb{R}$ is a free abelian group and find a basis.
3. Find an explicit, noninductive formula for the barycentric subdivision operator $S : C_n(X) \to C_n(X)$.

HW problems to hand in

1. (a) Verify that homotopy is an equivalence relation between continuous maps $X \to Y$.

   (b) Verify that chain homotopy of chain maps is an equivalence relation.

2. Prove the "5-lemma":

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow g_1 & & \downarrow g_2 & & \downarrow g_3 & & \downarrow g_4 & & \downarrow g_5 \\ B_1 & \xrightarrow{h_1} & B_2 & \xrightarrow{h_2} & B_3 & \xrightarrow{h_3} & B_4 & \xrightarrow{h_4} & B_5 \end{array}$$

   be a commutative diagram in of abelian groups, where the rows are exact sequences. If g_1, g_2, g_4, and g_5 are isomorphisms, then g_3 is also an isomorphism.

3. Let p be a prime number. Determine all isomorphism classes of abelian groups A that can appear as the middle term of a short exact sequence

$$0 \to \mathbb{Z}/(p^a) \to A \to \mathbb{Z}/(p^b) \to 0$$

4. (a) Prove that if each $x_i \in X_i$ has a neighbourhood that can be contracted to x_i and $\{x_i\} \subset X_i$ is a closed set, then

$$\widetilde{H}_k\left(\bigvee_{i=1}^n X_i\right) = \bigoplus_{i=1}^n \widetilde{H}_k(X_i)$$

   where $\bigvee_{i=1}^n X_i$ is the wedge sum of the spaces X_i identified at the points x_i.

   (b) Construct a topological space X such that for all $k \geq 0$,

$$H_k(X) = \mathbb{Z}^{n_k}$$

   where $n_k \in \mathbb{N}$ are arbitrary.

5. If X retracts onto A, prove that $H_n(X) \simeq H_n(A) \oplus H_n(X, A)$ for all $n \geq 0$.