Algebraic topology and homological algebra
Sheet 3 — Due 15/04

Practice problems (not to hand in)

1. Compute the cup product structure in $H^*(M_2)$ for M_2 the closed orientable surface of genus 2 (you can assume the cup product structure on the torus $S^1 \times S^1$).
2. Prove that $S^1 \vee S^1 \vee S^2$ and T^2 have the same homology but different cohomology rings.

HW problems to hand in

1. For each $d \in \mathbb{Z}$ and each $n \in \{1, \dots\}$, describe a surjective map $S^n \to S^n$ of degree d—pay particular attention to $d = 0$.
2. Given finitely generated abelian groups $A_1, \dots, A_n$ construct a space with

$$H_k(X) = \begin{cases} \mathbb{Z} \text{ if } k = 0 \\ A_k \text{ if } 0 \leq k \leq n \\ 0 \text{ otherwise} \end{cases}$$

   Hint: CW-complex

3. Produce a CW complex as follows. Start with a solid cube, with the polyhedral CW structure: there are eight 0-cells at the vertices, twelve 1-cells, being the edges, six 2-cells being the faces and a single interior 3-cell. Now form the quotient CW complex Q by identifying antipodal (closed) faces, but with a twist. For each pair of antipodal closed faces T and B, rotate the cube so T is on top and B on the bottom. Then identify T with B by twisting T by $\pi/2$ in the positive direction, and then projecting downwards. (Observe that the identification specified by this "anticlockwise twist" is the same whether T or B is placed on top.) The resulting CW complex Q has one 3-cell, three 2-cells and some number of 1- and 0-cells. Determine the cell structure of Q and calculate $H_*(Q)$.
4. Recall from class (or a differential topology course), that if M is an m-dimensional manifold and N is an n-dimensional submanifold of it (both oriented and compact), there exists a cohomology class $\omega_N \in H^{m-n}(N)$ that counts (with signs) the intersections with N. Assuming this, compute the cup product to deduce
   (a) $H^*(\mathbb{C}P^n) \simeq \mathbb{Z}[x]/(x^{n+1})$.
   (b) $H^*(\mathbb{R}P^n, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)[x]/(x^{n+1})$.

   Here the (co)homology with $\mathbb{Z}/2$ coefficients is obtained by replacing $\mathbb{Z}$ (resp. $\mathrm{Hom}(-, \mathbb{Z})$) with $\mathbb{Z}/2$ (resp. $\mathrm{Hom}(-, \mathbb{Z}/2)$) in the definitions. Don't mind that $\mathbb{R}P^n$ is non-orientable; the class ω_N also exists in this case, but in $H^{m-n}(N, \mathbb{Z}/2)$.
5. Show (e.g. by using Mayer-Vietoris) that $H^*(X \vee Y)$ is a subring of $H^*(X) \times H^*(Y)$. Describe this subring.