Algebraic topology and homological algebra
Sheet 4 — Due 29/04

Practice problems (not to hand in)

1. What is the product and coproduct in the category Top (= topological spaces and continuous maps)?
2. (a) Show that in the category of rings (with unit) the map $\mathbb{Z} \to \mathbb{Q}$ is epimorphism, but is not surjective.
   (b) Show that $\mathbb{Q}$, as an object of Ab, is not projective.
3. Show that for any $N \in R-\text{Mod}$:
   (a) $X \mapsto X \otimes N$ is a covariant right exact functor;
   (b) $X \mapsto \text{Hom}(N, X)$ is a covariant left exat functor;
   (c) $X \mapsto \text{Hom}(X, N)$ is a contravariant left functor;
   (d) for every $M, P \in R-\text{Mod}$ there exists a canonical isomorphism

   $$\text{Hom}(M \otimes N, P) \cong \text{Hom}(M, \text{Hom}(N, P))$$

   of R-modules. This property says that $X \mapsto X \otimes N$ and $X \mapsto \text{Hom}(N, X)$ are *adjoint* functors.

HW problems to hand in

1. (a) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as Abelian groups.
   (b) Show that the regular module $\mathbb{Z}/(n)$ is injective. Hint: Use Baer's criterion.
2. Show for the tensor product of complexes the following claims from the class
   (a) $\partial \circ \partial = 0$
   (b) $Z_i \otimes \widetilde{Z}_j \subseteq Z_{i+j}(C. \otimes \widetilde{C}.)$
   (c) $Z_i \otimes \widetilde{B}_j \subseteq B_{i+j}(C. \otimes \widetilde{C}.)$
3. Show that the followings are equivalent for an additive covariant functor:
   (a) $0 \to X \to Y \to Z \to 0$ exact $\Rightarrow F(X) \to F(Y) \to F(Z) \to 0$ exact
   (b) $X \to Y \to Z \to 0$ exact $\Rightarrow F(X) \to F(Y) \to F(Z) \to 0$ exact
4. (a) Compute $\text{Ext}^1(\mathbb{Z}/a, \mathbb{Z}/b)$.
   (b) Compute $\text{Tor}^1(\mathbb{Z}/a, \mathbb{Z}/b)$.
   (c) Show that for any ring R, $\text{Ext}^1(R/d, R) = R/d$.
   (d) Show that if u is not a zero divisor in R, then $\text{Tor}_k(R/u, M)$ is M/uM if $k = 0$ (this is called the u-torsion part of M), and 0 otherwise.
   (e) Convince yourself that $\text{Tor}_k(A, B) = \text{Tor}_k(B, A)$ (you don't need to hand this in).

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