Algebraic topology and homological algebra
Sheet 1 — Due 25/02

Practice problems (not to hand in)

1. (Partition of unity) Recall that a topological space X is called
   - *Hausdorff*, if for each pair of distinct points $x, y \in X$, there exist disjoint open subset U and V such that $x \in U$ and $y \in V$;
   - *compact* if every open cover of X has a finite subcover.

   Let X be a compact Hausdorff space and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Show that there exist a finite number of continuous real-valued functions $h_1, \dots, h_m$ on X with the following properties:

   (i) $0 \leq h_j \leq 1$, $1 \leq j \leq m$

   (ii) $\sum h_j = 1$

   (iii) For each $1 \leq j \leq m$, there is an index α_j such that the closure of the set $\{x : h_j(x) > 0\}$ is contained in U_{α_j}.

2. Prove, in every category, that each object has a unique identity morphism.

HW problems to hand in

1. For $n \geq 1$, define $\mathbb{R}P^n = S^n / \sim$, where the equivalence relation is defined by declaring $x \sim y$ if and only if $x = y$ or $x = -y$. In other words, $\mathbb{R}P^n$ is obtained by identifying pairs of antipodal points. The space $\mathbb{R}P^n$ is called the *real projective space* of dimension n, and it can be regarded as the set of lines in $\mathbb{R}^{n+1}$ which pass through the origin. Establish the following assertions:

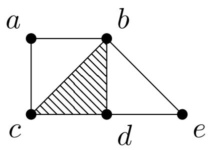
   (i) $\mathbb{R}P^n$ is compact and Hausdorff

   (ii) The projection $\pi : S^n \to \mathbb{R}P^n$ is a local homemorphism, that is, each $x \in S^n$ has an open neighbourhood that is mapped homeomorphically by π onto an open neighbourhood of $\pi(x)$.

   (iii) $\mathbb{R}P^1$ is homeomorphic to the circle S^1

   (iv) $\mathbb{R}P^n$ is homeomorphic to the quotient space obtained from the closed unit ball D^n in $\mathbb{R}^n$ by identifying antipodal points of its boundary S^{n-1}.

2. Compute the simplicial homology of the following simplicial complex:



3. Prove that $\Delta^n \approx D^n$ are homeomorphic.

4. (i) Prove that if f , g are composable morphisms in a category such that $g \circ f$ and g are isomorphisms, then f is an isomorphism.

   (ii) Let X be a space. Show that the assignment $Y \to X \times Y$ defines a functor $\text{Top} \xrightarrow{X \times} \text{Top}$.