Properties of the Ring of Formal Power Series

1. $R[[x]]$ is a Commutative Ring with 1

Define addition and multiplication in $R[[x]]$ as follows:

- **Addition:** Given two formal power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n,$$

their sum is defined as

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n.$$

This operation is associative and commutative because addition in R is associative and commutative.

- **Multiplication:** The product is defined as

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n,$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

This is the Cauchy product of the two sequences, and it follows that multiplication is associative and commutative because R is commutative.

- **Distributivity** follows directly from the properties of multiplication and addition in R.

- **Zero Element:** The power series with all coefficients zero, $0 = \sum_{n=0}^{\infty} 0x^n$, serves as the additive identity.

- **Multiplicative Identity:** The series $1 = 1 + 0x + 0x^2 + \dots$ serves as the multiplicative identity.

Thus, $R[[x]]$ is a commutative ring with 1.

2. $(1 - x)$ is a Unit in $R[[x]]$

To show that $(1 - x)$ is a unit, we must find a power series $g(x)$ such that

$$(1 - x)g(x) = 1.$$

We claim that

$$g(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

is the inverse of $1-x$. Multiplying formally,

$$(1-x)\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} x^{n+1}.$$

This is a telescoping sum, leaving 1. Thus,

$$(1-x)\sum_{n=0}^{\infty} x^n = 1.$$

Hence, $1-x$ is a unit in $R[[x]]$.

3. A Power Series is a Unit iff its Constant Term is a Unit in R

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. We need to determine when $f(x)$ is a unit in $R[[x]]$.

Necessity

If $f(x)$ is a unit, there exists $g(x) = \sum_{n=0}^{\infty} b_n x^n$ such that

$$f(x)g(x) = 1.$$

Comparing constant terms, we get

$$a_0 b_0 = 1.$$

Since $b_0 \in R$, this means a_0 must be a unit in R.

Sufficiency

If a_0 is a unit in R, we construct an inverse $g(x)$. Define

$$b_0 = a_0^{-1}.$$

For $n \geq 1$, we require

$$a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = 0.$$

Since a_0 is a unit, we solve recursively:

$$b_n = -a_0^{-1}(a_1 b_{n-1} + \cdots + a_n b_0).$$

This defines a unique power series $g(x)$, proving $f(x)$ is a unit.

4. If R is an Integral Domain, then so is $R[[x]]$

To show $R[[x]]$ is an integral domain, suppose $f(x)g(x) = 0$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Their product is

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n,$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

If $c_n = 0$ for all n, assume $f(x) \neq 0$ and $g(x) \neq 0$. Let a_r be the first nonzero coefficient in $f(x)$, and b_s be the first nonzero coefficient in $g(x)$. Then, the coefficient of x^{r+s} in the product is

$$c_{r+s} = a_r b_s + \text{(higher order terms)}.$$

Since $a_r, b_s \neq 0$ and R is an integral domain, their product $a_r b_s \neq 0$, contradicting $c_n = 0$ for all n. Thus, at least one of $f(x)$ or $g(x)$ must be zero, proving $R[[x]]$ is an integral domain.