Calculus and Diffy Q's

Ádám Gyenge

Basic Differentiation

SymPy can perform differentiation of mathematical expressions symbolically. This is useful for obtaining exact representations of derivatives.

```
# Define symbols and an expression
x, y = sp.symbols('x y')
expression = x**2 + sp.sin(x*y)

# Compute the partial derivative with respect to x
derivative_x = sp.diff(expression, x)

# Compute the partial derivative with respect to y
derivative_y = sp.diff(expression, y)
```

Basic Differentiation Result

The partial derivative with respect to x is $2x + y$, and the partial derivative with respect to y is $x \cos(xy)$.

```
>>> derivative_x
2*x + y*cos(x*y)

>>> derivative_y
x*cos(x*y)
```

Higher-Order Differentiation

For higher-order derivatives, you can either pass the variable multiple times or use the number of derivatives. Example:

```
>>> sp.diff(x**5, x, x)
20*x**3

>>> sp.diff(x**5, x, 2)
20*x**3
```

Mixed Partial Derivatives

SymPy also computes mixed partial derivatives.

```
>>> x, y, z = sp.symbols('x y z')
>>> sp.diff(sp.exp(x*y*z), x, y, y, z, z)

x*(x**3*y**3*z**3 + 8*x**2*y**2*z**2 + 14*x*y*z + 4)
    *exp(x*y*z)
```

Symmetry of second derivatives

Theorem (Young's theorem)

Exchanging the order of partial derivatives of a twice-differentiable multivariate function

$$f(x_1, x_2, \ldots, x_n)$$

does not change the result. That is, the second-order partial derivatives satisfy the identities

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right).$$

Implicit functions

- An *implicit function* is defined by an equation of the form $F(x, y) = 0$, where F is a function of two variables, and y is implicitly related to x.
- Unlike explicit functions, where y is expressed directly in terms of x, implicit functions require solving the equation $F(x, y) = 0$ to determine y as a function of x.
- In many cases, it is challenging or impossible to express y explicitly, yet the implicit function theorem guarantees the existence of y as a differentiable function of x under certain conditions.

Implicit Differentiation

To differentiate an implicit function, we apply implicit differentiation. Taking the total derivative of $F(x, y)$ with respect to x, we use the chain rule:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

Rearranging, we solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}},$$

provided that $\frac{\partial F}{\partial y} \neq 0$. This allows us to compute the derivative without explicitly solving for y.

Implicit Differentiation

To differentiate implicit functions, use SymPy's `idiff()` function.
Example:

```
# Define an implicit equation
implicit_eq = x**5 + y**2 + z**4 - 8*x*y*z

# Differentiate implicitly
implicit_derivative = sp.idiff(implicit_eq, y, x)
```

Implicit Differentiation Result

The derivative of y with respect to x is:

```
>>> implicit_derivative
(5*x**4/2 - 4*y*z)/(4*x*z - y)

>>> implicit_derivative.simplify()
(5*x**4 - 8*y*z)/(2*(4*x*z - y))
```

Unevaluated Derivatives

You can also create unevaluated derivatives using the `Derivative` class. Example:

```
# Create an unevaluated derivative
deriv = sp.Derivative(sp.exp(x*y*z), x, y, 2, z, 2)
```

Result:

```
>>> deriv
Derivative(exp(x*y*z), x, (y, 2), (z, 2))
```

We get nicer result if we initialize pretty printing:

```
>>> sp.init_printing()
>>> deriv
     5
    ∂        ⎛ x⋅y⋅z⎞
─────────────⎝e     ⎠
  2    2
∂z   ∂y   ∂x
```

Unevaluated Derivative

- Unevalutated derivatives can be useful when we want to perform complex operations.
- To evaluate the derivative, use the `doit()` function:

```
>>> deriv.doit()
x*(x**3*y**3*z**3 + 8*x**2*y**2*z**2 + 14*x*y*z + 4)
   *exp(x*y*z)
```

Antiderivatives

SymPy supports symbolic integration. Here's an example of computing an indefinite integral:

```
# Define a simple expression
expression = x**2 + sp.sin(x)

# Compute the indefinite integral
indefinite_integral = sp.integrate(expression, x)
```

Result:

```
>>> indefinite_integral
x**3/3 - cos(x)
```

Definite Integration

For definite integrals, you can specify the limits:

```
# Define the limits of integration
lower_limit = 0
upper_limit = sp.pi

# Compute the definite integral
definite_integral =
    sp.integrate(expression, (x, lower_limit, upper_limit))
```

The result of the definite integral from 0 to π is:

```
>>> definite_integral
2 + pi**3/3
```

Improper Integrals

To compute improper integrals, use oo to denote infinity:

```
# Define an improper integral
improper_integral = sp.integrate(sp.exp(-2*x),
    (x, 0, sp.oo))
```

The result of the improper integral is:

```
>>> improper_integral
1/2
```

Integration of Special Functions

SymPy supports the integration of functions like $\sin(x)$ and e^x:

```
# Define an expression with special functions
special_expression = sp.sin(x) * sp.exp(x)

# Compute the integral
integral_special = sp.integrate(special_expression, x)
```

The integral of $\sin(x) \cdot e^x$ is:

```
>>> integral_special
exp(x)*sin(x)/2 - exp(x)*cos(x)/2
```

Multiple integrals

To compute

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\mathrm{d}x\mathrm{d}x$$

one can type the following:

```
# Define a multiple integral
multiple_integral = integrate(sp.exp(-x**2 - y**2),
    (x, -sp.oo, sp.oo), (y, -sp.oo, sp.oo))
```

The result will be π.

```
>>> multiple_integral
pi
```

Unevaluated Integrals

As with derivatives, we can create unevaluated integrals:

```
# Define unevaluated integral
expr = sp.Integral(x**2, x)
```

This gives:

```
>>> expr
Integral(x**2, x)

>>> expr.doit()
x**3/3
```

Limit Computations

To compute limits, use the `limit()` function. Example:

```
# Define the variable and expression
expression = sp.sin(x) / x

# Compute the limit
limit_result = sp.limit(expression, x, 0)
```

The limit as $x \to 0$ of $\frac{\sin(x)}{x}$ is:

```
>>> limit_result
1
```

One-Sided Limits

SymPy can compute one-sided limits using the `dir` parameter.
Example:

```
# Define the piecewise function
f = sp.Piecewise((x**2, x >= 0), (-x**2, x < 0))

# Compute one-sided limits
limit_right = sp.limit(f, x, 0, dir='+')
limit_left = sp.limit(f, x, 0, dir='-')
```

The results for the right and left-hand limits are:

```
>>> limit_right
0

>>> limit_left
0
```

Multivariate Limits

SymPy can compute multivariate limits. Example:

```
# Define variables and the expression
x, y = sp.symbols('x y')
expression = (x**2 + y**2) / (x**2 + 2*y**2)

# Compute the multivariate limit
multivariate_limit = sp.limit(expression, x, 0, y, 0)
```

The result of the multivariate limit is:

```
>>> multivariate_limit
1/2
```

Taylor Series Expansion

SymPy can compute Taylor series expansions. Example:

```
# Define the variable and expression
expression = sp.sin(x)

# Compute the series expansion
taylor_series = sp.series(expression, x, 0, 6)
```

Here we take the expansion around 0 up to order 6.
Result:

```
>>> taylor_series
x - x**3/6 + x**5/120 + O(x**6)
```

Laurent Series

Laurent series expansions can also be computed. Example:

```
# Define the expression
expression = 1 / (x - 1)

# Compute the Laurent series
laurent_series = sp.series(expression, x, 0, 5)
```

The result of the Laurent series expansion is:

```
>>> laurent_series
-1/x + O(x)
```

Asymptotic expansion

Asymptotic expansion is possible at infinity.
For instance, to compute the asymptotic expansion of $\ln(1+x)$ as $x \to \infty$:

```
# Define the expression
expression = sp.log(1 + x)

# Compute the asymptotic expansion
asymptotic_series = sp.series(expression, x, sp.oo, 3)
```

The result is:

```
>>> asymptotic_series
log(x) + 1/x - 1/(2*x**2) + O(1/x**3)
```

Order terms and truncation

- Series expansions in SymPy include an $O()$ term, which represents higher-order terms.
- The `removeO()` method can be used to truncate the series by removing the $O()$ term.
- For example:

```
# Remove the order term
truncated_series = taylor_series.removeO()
```

The result is:

```
>>> truncated_series
x - x**3/6 + x**5/120
```

Defining Differential Equations

Symbolic Representation

- Use Eq objects to define equations.
- Declare variables and functions beforehand.

```
import sympy as sp

x = sp.symbols('x')
y = sp.Function('y')
eq = sp.Eq(sp.Derivative(y(x), x, x) + y(x), 0)
```

Example: $y'' + y = 0$

```
>>> eq
Eq(y(x) + Derivative(y(x), (x, 2)), 0)
```

Functions in SymPy

Using `Function` class

- Functions of one or more variables are represented using the class `Function`.

```
y = sp.symbols('y', cls=sp.Function)
```

- Functions allow symbolic differentiation and equation manipulation.

Importance of Declaring Dependencies

- $y(x)$ must be declared as a function of x for:
  - Derivatives
  - Symbolic manipulations

Solving Equations

General Solutions with `dsolve()`

- Use `dsolve()` to solve differential equations symbolically.

```
solution = sp.dsolve(eq, y(x))
```

Solution for $y'' + y = 0$:

```
>>> solution
Eq(y(x), C1*cos(x) + C2*sin(x))
```

Extracting and Manipulating Solutions

- Extract LHS and RHS:

```
>>> solution.lhs
y(x)

>>> solution.rhs
C1*sin(x) + C2*cos(x)
```

- Substitute constants:

```
>>> solution.rhs.subs({'C1': 1, 'C2': 2})
sin(x) + 2*cos(x)
```

Verifying Solutions

Using `checkodesol()`

- Verify that a solution satisfies the original equation:

```
check_result = sp.checkodesol(eq, solution)
```

Output:

```
>>> check_result
(True, 0)
```

Initial Conditions and Boundary Value Problems

Solving with Initial Conditions

- Example: Solve $y'' + y = 0$ with:
  - $y(0) = 1$
  - $y'(0) = 0$

```
solution_ic = sp.dsolve(eq, y(x), ics={y(0): 1,
    y(x).diff(x).subs(x, 0): 0})
```

Result:

```
>>> solution_ic
Eq(y(x), cos(x))
```

Plotting Solutions

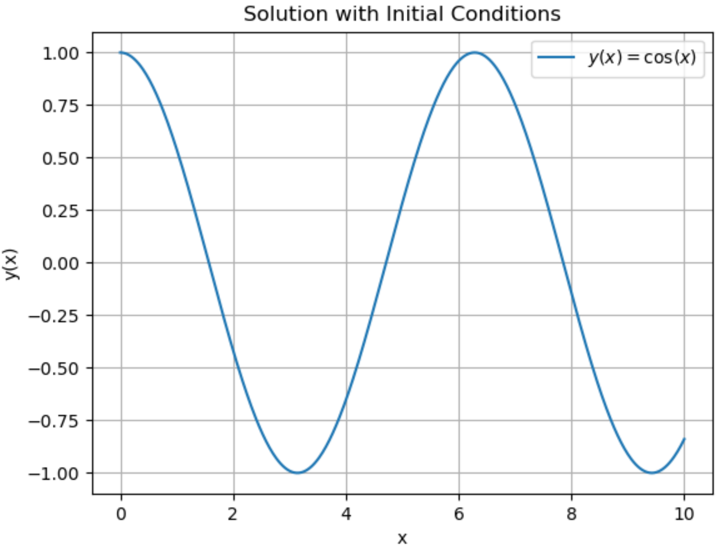
We can then plot using `lambdify`:

```
import numpy as np
import matplotlib.pyplot as plt

x_vals = np.linspace(0, 10, 500)
sol_func_ic = sp.lambdify(x, solution_ic.rhs, 'numpy')
y_vals = sol_func_ic(x_vals)

plt.plot(x_vals, y_vals, label='$y(x) = \cos(x)$')
plt.xlabel('x')
plt.ylabel('y(x)')
plt.title('Solution with Initial Conditions')
plt.legend()
plt.grid()
plt.show()
```

Plotting Solutions



Systems of ODEs

Solving Coupled Equations

- Example System:

$$y_1' = y_2, \quad y_2' = -y_1$$

```
y1, y2 = sp.symbols('y1 y2', cls=sp.Function)
eq1 = sp.Eq(y1(x).diff(x), y2(x))
eq2 = sp.Eq(y2(x).diff(x), -y1(x))
solution_system = sp.dsolve([eq1, eq2])
```

Solution:

```
>>> solution_system
[Eq(y1(x), C1*sin(x) + C2*cos(x)),
    Eq(y2(x), C1*cos(x) - C2*sin(x))]
```

Partial Differential Equations (PDEs)

Defining PDEs

- Example: First-order linear PDE:

$$2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 5$$

```
x, y = sp.symbols('x y')
u = sp.Function('u')(x, y)
pde = sp.Eq(2 * sp.Derivative(u, x)
    + 3 * sp.Derivative(u, y), 5)
```

Partial Differential Equations (PDEs)

Solving PDEs

- Use `pdsolve()`:

```
solution = sp.pdsolve(pde)
```

General solution:

```
>>> solution
Eq(u(x, y), 10*x/13 + 15*y/13 + F(3*x - 2*y))
```

Here F is an unknown function.

Adding Initial Conditions to PDEs

- Example: Given $u(x,0) = x^2$:

```
u0 = x**2
specific_condition = solution.subs(y, 0).rhs - u0
F_general = sp.simplify(specific_condition)
```

Particular solution:

```
>>> F_general
-x**2 + 10*x/13 + F(3*x)
```

The meaning of this is that

$$-x^2 + \frac{10x}{13} + F(3x) = 0.$$

We can then substitute F back into the general solution to obtain the particular solution.