Computational Topology

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Outline

Reference:

1. Edelsbrunner-Harer: Computational topology: an introduction.

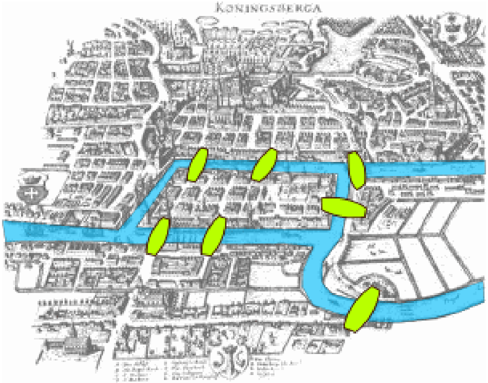
Topology

From Wikipedia:

Topology (from the Greek words $\tau o \pi o o \sigma$, 'place, location', and $\lambda o \gamma o \sigma$, 'study') is the branch of mathematics concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling, and bending; that is, without closing holes, opening holes, tearing, gluing, or passing through itself.

Origins

Euler (1736): Seven bridges of Königsberg



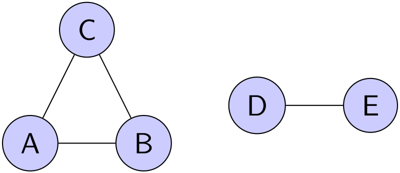
$$V - E + F = 2$$

Graphs and Paths

- A **graph** G consists of vertices (nodes) V and edges (connections) E.
- **Simple graph:** no two edges connect the same two vertices and no edge joins a vertex to itself
- A **path** in a graph is a sequence of edges connecting a sequence of vertices.
- A graph is **connected** if there is a path between any two vertices, otherwise it is **disconnected**
- A **(connected) component** is a maximal subgraph that is connected.

Separation and Connectedness

- A **separation** is a non-trivial partition of the vertices, that is, $V = U \sqcup W$ with $U, W \neq \emptyset$, such that no edge connects a vertex in U with a vertex in W.
- A simple graph is connected if it has no separation.



Topological Spaces

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X, called *open sets*, such that

1. X is open and the empty set $\emptyset$ is open;
2. if U_1 and U_2 are open, then $U_1 \cap U_2$ is open;
3. if U_i is open for all i in some possibly infinite, possible uncountable, index set, then the union of all U_i is open.

Remarks

- The pair $(X, \mathcal{T})$ is a called a **topological space**, but we will usually tacitly assume that $\mathcal{T}$ is understood and refer to X a topological space.
- Condition (ii) is equivalent to requiring that intersections of finitely many open sets are open.
- **closed sets** $=$ complement of open sets

Closure, Interior, and Boundary of a Subset in a Topological Space

- Let $(X, \mathcal{T})$ be a **topological space** and $A \subseteq X$ a subset.

Definitions

- **Closure** of A, denoted $\overline{A}$: the smallest closed set containing A.

$$\overline{A} = \bigcap \{U \in \mathcal{T} : A \subseteq U\}.$$

- **Interior** of A, denoted $\text{int}(A)$: the largest open set contained in A.

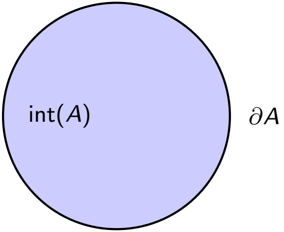
$$\text{int}(A) = \bigcup \{U \in \mathcal{T} : U \subseteq A\}.$$

- **Boundary** of A, denoted ∂A:

$$\partial A = \overline{A} \setminus \text{int}(A).$$

Examples in $\mathbb{R}^2$ with the standard topology

- Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ (open unit disk):
    - $\text{int}(A) = A$.
    - $\overline{A} = \{(x, y) : x^2 + y^2 \leq 1\}$ (closed unit disk).
    - $\partial A = \{(x, y) : x^2 + y^2 = 1\}$ (circle).



$\overline{A}$ (shaded region)

Topological Basis

- A **basis** of a topology on a point set X is a collection $\mathcal{B}$ of subsets of X, called basis elements, such that each $x \in X$ is contained in at least one $B \in \mathcal{B}$ and $x \in B_1 \cap B_2$ implies there is a third basis element with $x \in B_3 \subseteq B_1 \cap B_2$.
- The topology $\mathcal{T}$ **generated** by $\mathcal{B}$ consists of all sets $U \subseteq X$ for which $x \in U$ implies there is a basis element $x \in B \subseteq U$. This topology can be constructed explicitly by taking all possible unions of all possible finite intersections of basis sets.
- Example: The standard topology on $\mathbb{R}$ with basis as open intervals (a, b).

Examples of Topological Spaces

- **The Real Line with the Standard Topology:**
  - The set of real numbers $\mathbb{R}$, with the standard topology (open intervals), is a topological space.
- **The Sierpiński Space:**
  - A topological space with two points $X = \{a, b\}$, and the topology $\{\emptyset, X, \{a\}\}$, where $\{a\}$ is open.
- **Discrete Topology:**
  - Every subset is open.

Example: Metric Spaces

- A **metric space** is a set M equipped with a distance function $d : M \times M \to \mathbb{R}$ satisfying:
  1. $d(x, y) \geq 0$ with $d(x, y) = 0 \iff x = y$ (**non-negativity**),
  2. $d(x, y) = d(y, x)$ (**symmetry**),
  3. $d(x, z) \leq d(x, y) + d(y, z)$ (**triangle inequality**).
- A metric space induces a **topology** via the basis of open balls:

$$B_r(p) = \{x \in M : d(p, x) < r\}, \quad r > 0,\ p \in M.$$

- The open sets in this topology are unions of open balls:

$$U \subseteq M \text{ is open if } \forall x \in U, \exists r > 0 \text{ such that } B_r(x) \subseteq U.$$

Examples

- $\mathbb{R}^n$ with $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ (Euclidean metric).
- Discrete metric: $d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$

Product Topology

- Let $X = \prod_{i \in I} X_i$ be the product of topological spaces $\{X_i\}_{i \in I}$, where I is an index set.
- The **product topology** on X is defined by a basis of open sets.

Basis for the Product Topology

- A basis for the product topology consists of all sets of the form:

$$\prod_{i \in I} U_i$$

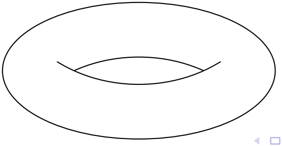
  where $U_i \subseteq X_i$ is open, and $U_i = X_i$ for all but finitely many $i \in I$.
- In other words, an open set in the product topology is a product of open sets in each X_i, but only finitely many U_i are non-trivial.

Examples of Product Spaces 1.

- Let $X_1 = \mathbb{R}$ and $X_2 = \mathbb{R}$. The product space $X_1 \times X_2 = \mathbb{R}^2$ is endowed with the product topology.
- A basis element in $\mathbb{R}^2$ is a set of the form $U_1 \times U_2$, where $U_1 \subseteq \mathbb{R}$ and $U_2 \subseteq \mathbb{R}$ are open sets.
- **Torus:** The torus T^2 is the product of two circles:

$$T^2 = S^1 \times S^1$$

where $S^1 = \mathbb{R}/\mathbb{Z}$ is the unit circle, i.e., the real numbers modulo 1. The product topology on T^2 is the topology inherited from the product of two copies of S^1, and it represents a 2-dimensional surface that can be visualized as a doughnut shape.



Examples of Product Spaces 2.

Cylinder: The cylinder C can be represented as a product space:

$$C = S^1 \times \mathbb{R}$$

where S^1 is the unit circle and $\mathbb{R}$ is the real line. The cylinder is a 2-dimensional surface that extends infinitely in one direction, with circular cross-sections.

Subspace Topology

- Let X be a topological space and $Y \subseteq X$.
- The **subspace topology** on Y is defined by $U \cap Y$ for $U \in \mathcal{T}$, where $\mathcal{T}$ is the topology of X.
- Example: Subspace topology of $[0, 1] \subset \mathbb{R}$ inherits open intervals from $\mathbb{R}$.

Continuous Maps Between Metric Spaces

Let $(X, d_X), (Y, d_Y)$ be metric spaces.

Definition

A map $f : (X, d_X) \to (Y, d_Y)$ is **continuous** if $\forall \epsilon > 0,\ \exists \delta > 0$, such that for all $x_1, x_2 \in X$, if $d_X(x_1, x_2) < \delta$, then $d_Y(f(x_1), f(x_2)) < \epsilon$.

Proposition

A map f is continuous if the preimage of every open subset in Y is open in X:

$$\forall U \subseteq Y, \text{ if } U \text{ is open, then } f^{-1}(U) \text{ is open in } X$$

Continuity

Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces.

Definition

A function $f : X \to Y$ is **continuous** if the preimage of every open set in Y is open in X.

Example: $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$ is continuous.

Paths in Topological Spaces

- A **path** in a topological space is a continuous map $\gamma : [0, 1] \to X$.
- A space is **path-connected** if any two points can be connected by a path.
- Examples of path-connected spaces: $\mathbb{R}^n$, circles.

Disjoint Systems

A **separation** of a topological space X is a partition $X = U \sqcup W$ into two non-empty, open subsets. A topological space is **connected** if it has no separation.

Proposition

X path-connected $\Rightarrow$ X connected

- A **disjoint system** is a collection of disjoint connected components.
- Useful for simplifying computations in connectedness analysis.
- Example: Forest of trees in a disconnected graph.

Disjoint-Set data structure

Given a graph, we want to find connected components algorithmically.

- Vertices: integers from 1 to n
- Components of the graph: subsets of $[n] = \{1, \ldots, n\}$

Idea:

1. Add the edges one at a time and maintain the system of sets representing the components
2. We find that the graph is connected iff in the end there is only one set left, namely $[n]$

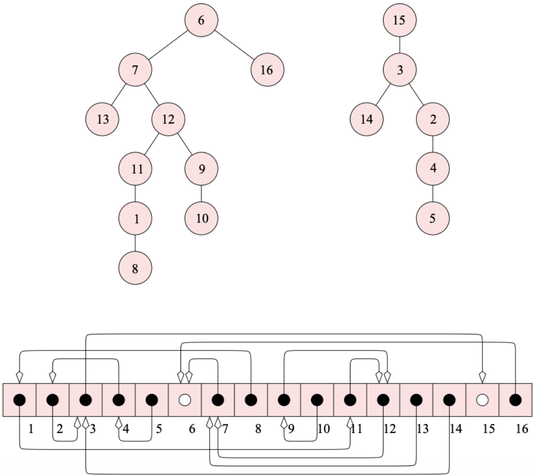
Disjoint-Set data structure

The Disjoint-Set data structure is used to manage a partition of a set into disjoint subsets. It supports two main operations:

- **Find(x)**: Returns the representative or leader of the set containing element x.
- **Union(x, y)**: Merges the sets containing x and y.

A standard data structure implementing a disjoint set system stores each set as a tree embedded in a linear array, $V[1, \ldots, n]$. Each node in the tree is equipped with a pointer to its parent, except for the root which has no parent.

Example



Pseudocode: Find Operation

```
Find(x):
    if parent[x] != null:
        return Find(parent[x])
    else
        return x
```

If i is not the root then we find the root recursively and finally return it. Otherwise, we return i as the root.

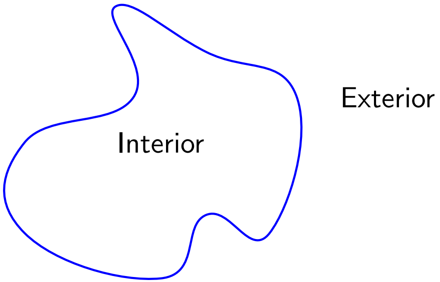
Pseudocode: Union Operation

```
Union(x, y):
    rootX = Find(x)
    rootY = Find(y)
    if rootX != rootY:
            parent[rootX] = rootY
```

After making sure that the two sets are different, we assign one root as the parent of the other.

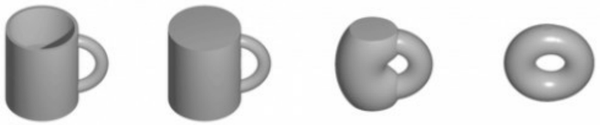
Closed Curves

- A **closed curve** is a continuous function $\gamma : [0,1] \to \mathbb{R}^2$ with $\gamma(0) = \gamma(1)$.
- Equivalently, it is a map from the unit circle, $\gamma : S^1 \to X$, where $S^1 = x \in \mathbb{R}^2 : |x| = 1$.
- Examples:
    - Circle: $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$.
    - Ellipse: $\gamma(t) = (a\cos(2\pi t), b\sin(2\pi t))$.



Homeomorphisms

We call two topological spaces **homeomorphic** or **topologically equivalent** if there exists a continuous bijection from one space to the other whose inverse is also continuous.
A map with these properties is called a **homeomorphism**.



Claim

The unit interval and the unit circle are not homeomorphic.

Proof.

Removing the midpoint decomposes the interval into two components while removing its image leaves the circle connected. This contradicts the existence of a bijection that is continuous in both directions. □

Jordan Curve Theorem I

A **simple closed curve** is a curve without self-intersections, that is, a continuous injection $\gamma : S^1 \to \mathbb{R}^2$.

Theorem (Jordan Curve Theorem)

Removing the image of a simple closed curve from $\mathbb{R}^2$ *leaves two connected component:*

- **Interior (inside)**: *A bounded region.*
- **Exterior (outside)**: *An unbounded region.*

The inside together with the image of the curve is homeomorphic to a closed disk.

Corollary

Any continuous path connecting points from the interior and exterior must cross the curve.

Jordan Curve Theorem II

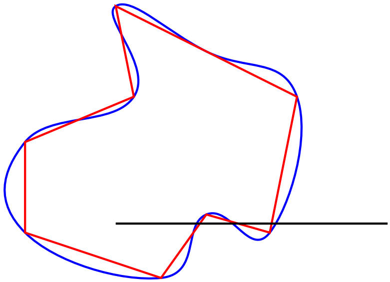
- Remains valid if we replace the plane by the sphere, $S^2 = \{x \in \mathbb{R}^3 : ||x|| = 1\}$, but not if we replace it by the torus.
- Provides a foundation for:
  - Topological classification of planar regions.
  - Algorithms for point-in-polygon tests.
- Applications in computer graphics and geometric modeling.

Parity Algorithm

- Used to determine if a point lies inside a closed curve.
- Steps:
  1. Draw a ray from the point in any direction.
  2. Count the number of intersections between the ray and the curve.
  3. **Odd**: Point is inside. **Even**: Point is outside.
- Example:
  - A point inside a triangle will have one intersection.
  - A point outside a triangle will have zero or an even number of intersections.

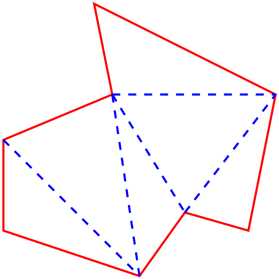
Example

In practice, we use a polygonal approximation: we specify γ at a finite number of points and interpolate linearly between them $\rightarrow$ get a *closed polygon*



Polygon Triangulation

- A **polygon triangulation** divides a simple polygon into triangles by drawing non-intersecting *diagonals*.
- Properties:
  - A polygon with n vertices can always be triangulated into $n-2$ triangles.
  - Computational complexity: $O(n \log n)$ for simple polygons.
- Triangulation is used in computer graphics, finite element methods, and computational geometry.



Existence of triangulartions

We need to show that there is at least one diagonal, unless the number of edges in the polygon is n $= 3$. We may consider the leftmost vertex, b, of the polygon.

- Either we can connect its two neighbors, a and c
- Or we can connect b to the leftmost vertex u that lies inside the triangular region abc. Drawing this diagonal decomposes the n-gon into two, an n_1-gon and an n_2-gon. We have $n_1 + n_2 = n + 2$ and since both are at least three, we also have $n_1, n_2 < n$.

Then use induction.

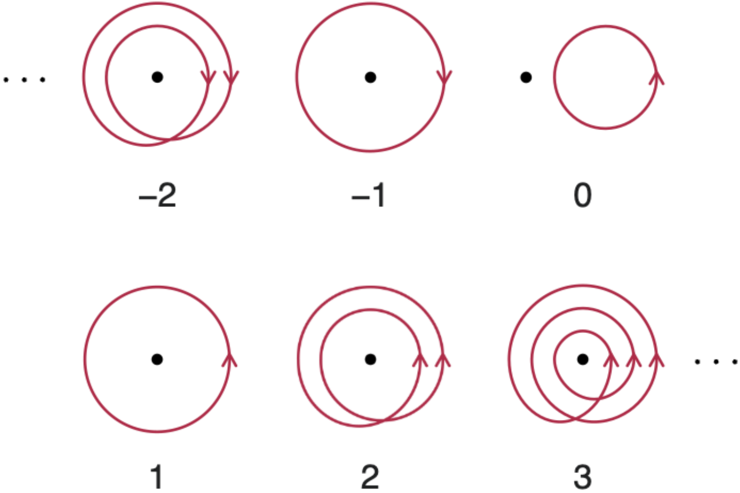
Winding Number

- Measures how many times a curve winds around a given point.
- Definition:

$$w(\gamma, p) = \frac{1}{2\pi} \int_\gamma \frac{(x - p_x)dy - (y - p_y)dx}{(x - p_x)^2 + (y - p_y)^2}$$

- Properties:
  - $w(\gamma, p) = 0$: Point lies outside.
  - $w(\gamma, p) \neq 0$: Point lies inside.

Visualizing Winding Number



Applications of Winding Number

- Determining interior/exterior of polygons.
- Robotics: Path planning around obstacles.
- Computer graphics: Texture mapping and clipping.