Two-Dimensional Manifolds

Ádám Gyenge

What is a 2-Manifold?

- A **2-manifold** (or surface) is a topological space M such that for every $p \in M$, there exists a neighborhood $U \subseteq M$ and a homeomorphism $\phi : U \to \mathbb{R}^2$.
- Intuitively, M locally looks like $\mathbb{R}^2$ but may have a more complex global structure.

Examples

- Sphere: $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$,

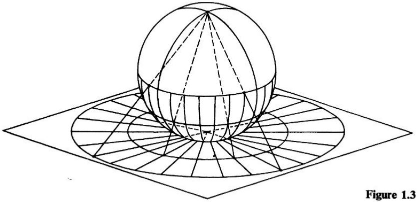
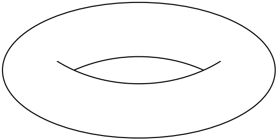


Figure 1.3

- Torus: $T^2 = S^1 \times S^1$.

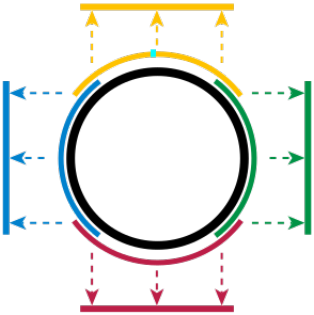


n-manifolds

Similarly, we can define n-manifolds: each point has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Example

1. The circle S^1 is a 1-manifold



2. The 3-sphere $S^3 = \mathbb{R}^2 \cup \infty$ is a 3-manifold.

Compactness

- A space M is **compact** if every open cover $\{U_\alpha\}_{\alpha \in A}$ of M has a finite subcover.
- Formal definition: $\exists\, \alpha_1, \alpha_2, \ldots, \alpha_n \in A$ such that $M \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.
- Compact 2-manifolds are closed (no boundary) and bounded.

Examples

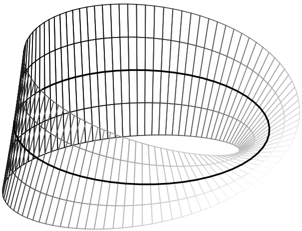
- Sphere S^2: Compact.
- Plane $\mathbb{R}^2$: Not compact.

2-Manifolds with Boundary

- A **2-manifold with boundary** is a surface where each point has a neighborhood homeomorphic to either:
  - $\mathbb{R}^2$ (interior points), or
  - $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ (boundary points, half-plane).
- The boundary ∂M is a (potentially empty) 1-manifold.

Examples

- Disk $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.
- Möbius strip: A non-orientable manifold with boundary.



Quotient Topology

- The **quotient topology** is a way to construct a new topological space by identifying points of an existing space according to an equivalence relation.
- Let (X, τ) be a topological space and $\sim$ an equivalence relation on X.
- The quotient space $X/\sim$ is the set of equivalence classes:

$$X/\sim = \{[x] : x \in X\}, \quad [x] = \{y \in X : y \sim x\}.$$

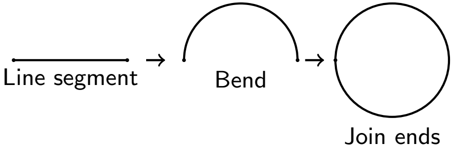
- The quotient topology on $X/\sim$ is defined as:

$$U \subseteq X/\sim \text{ is open if and only if } \pi^{-1}(U) \text{ is open in } X,$$

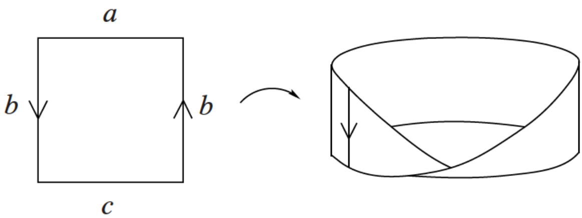
where $\pi : X \to X/\sim$ is the natural projection $\pi(x) = [x]$.

Examples

- **Circle from a Line Segment**: $[0, 1]$ with $0 \sim 1$ gives S^1.

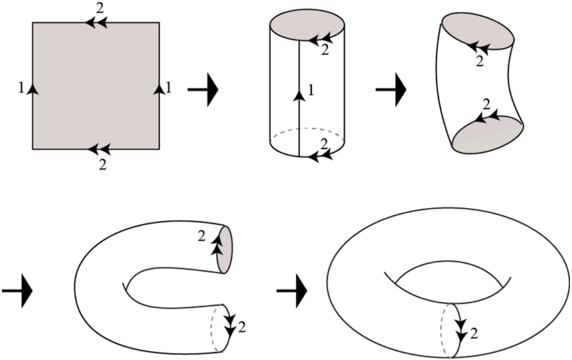


- **Möbius band from a Square**:



Examples

- **Torus from a Square**: Identify opposite edges of a square.



Polygon Construction

- Let P be a finite-sided convex polygon (in $\mathbb{R}^2$) with an *even* number of sides.
- The sides of the polygon P are arranged into pairs.
- Let e and e' be two sides of a pair.
- Suppose that e runs from (x_0, y_0) to (x_1, y_1) and e' runs from (x'_0, y'_0) to (x'_1, y'_1).
- As t runs from 0 to 1, the point

$$(1-t)(x_0, y_0) + t(x_1, y_1)$$

lies on e and the point

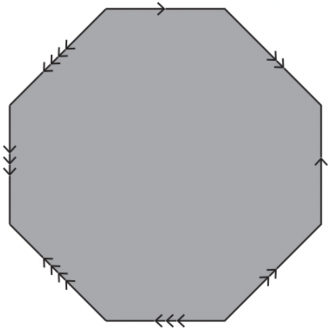
$$(1-t)(x'_0, y'_0) + t(x'_1, y'_1)$$

lies on e'.

Identification of Points on Paired Sides

- For each $t \in [0, 1]$, we identify the corresponding points on e and e'.
- There is still a choice regarding the orientation of the sides to be identified.
- We can choose for e to run from (x_0, y_0) to (x_1, y_1) or the other way around.
- We encode this choice by drawing an arrow on e, running from (x_0, y_0) to (x_1, y_1).
- Once arrows have been drawn on both e and e', this determines how the sides are identified.

Example



An octagon with side identifications

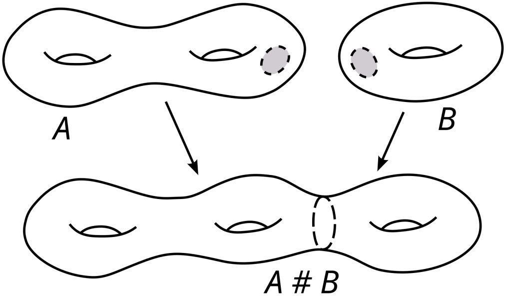
The two-holed torus

Connected Sum

- The **connected sum** $M \# N$ of two 2-manifolds M and N is constructed by removing a disk from each and gluing along the resulting boundaries.
- The genus g of the connected sum is additive:

$$g(M \# N) = g(M) + g(N).$$

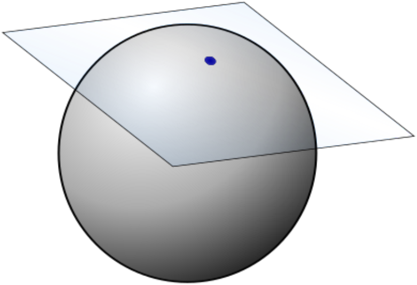
Example



Tangent Space of a Manifold

What is a Tangent Space?

- Let M be a smooth n-dimensional manifold. The tangent space at a point $p \in M$, denoted T_pM, is a vector space that intuitively represents the directions in which one can tangentially pass through p.
- It generalizes the concept of tangent lines and planes to arbitrary manifolds.



Tangent Space of a Manifold

Formal Definition:

- The tangent space T_pM is the set of equivalence classes of smooth curves passing through p.
- A smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ determines a tangent vector at p through its velocity:

$$v = \dot{\gamma}(0) \in T_pM.$$

Dimension of the Tangent Space:

- If M is an n-dimensional manifold, then $\dim(T_pM) = n$.
- In local coordinates $(x^1, \ldots, x^n)$, a basis for T_pM is given by:

$$\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}.$$

Tangent Space of a Manifold

Properties:

- The tangent space T_pM is a real vector space.
- It depends only on the local geometry of M near p.
- A smooth map $f : M \to N$ induces a linear map between tangent spaces:

$$f_* : T_pM \to T_{f(p)}N,$$

known as the **differential** or **pushforward**.

Visualization:

- For a surface $M \subset \mathbb{R}^3$, the tangent space T_pM is the plane that "touches" M at p and is spanned by the partial derivatives of the local parameterization.
- Example: For the sphere S^2, the tangent space T_pS^2 is the plane perpendicular to the radius vector at p.

Orientation of Vector Spaces

- A real vector space V of dimension n is said to be **oriented** if we choose an equivalence class of ordered bases.
- Two ordered bases $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ are in the same equivalence class if the change of basis matrix has $\det > 0$.
- If an orientation is chosen, the vector space is called an **oriented vector space**.

Determinants and Orientation

- The determinant of a basis transformation matrix determines the orientation:

$$\det(A) > 0 \Rightarrow \text{same orientation}$$

$$\det(A) < 0 \Rightarrow \text{opposite orientation}$$

- Example: In $\mathbb{R}^2$, the standard basis (e_1, e_2) and (e_2, e_1) have opposite orientations.
- Applications:
  - Oriented manifolds.
  - Integration on manifolds (Stokes' theorem).
  - Physics: Right-hand rule in 3D.

Orientability

- A manifold M is **orientable** if it admits a globally consistent choice of orientation for its tangent spaces.
- Equivalently, if every closed curve in M is orientation-preserving.

Examples

- Sphere S^2, torus T^2: Orientable.
- Möbius strip, Klein bottle: Non-orientable.

Orientation-Preserving and Reversing Curves

- A curve $\gamma : [0,1] \to M$ is **orientation-preserving** if it does not reverse the manifold's orientation.
- Example: A loop on the torus.
- A curve γ is **orientation-reversing** if it reverses orientation.
- Example: A closed curve on the Möbius strip.

Projective Plane

Definition

The (real) *projective plane*, denoted by $\mathbb{RP}^2$, is the set of all lines passing through the origin in $\mathbb{R}^3$. Formally:

$$\mathbb{RP}^2 = \{\ell \subset \mathbb{R}^3 \mid \ell \text{ is a 1-dimensional subspace of } \mathbb{R}^3\}.$$

Equivalent Construction:

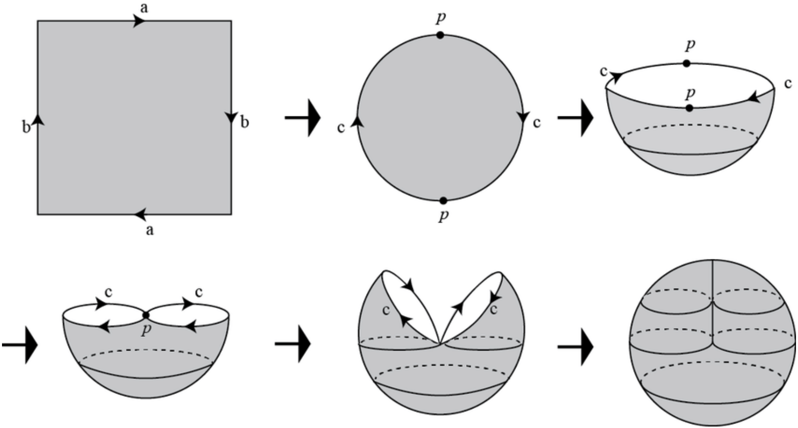
- $\mathbb{RP}^2$ can also be defined as the set of equivalence classes of $\mathbb{R}^3 \setminus \{0\}$ under the equivalence relation:

  $$\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} = \lambda \mathbf{y}, \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

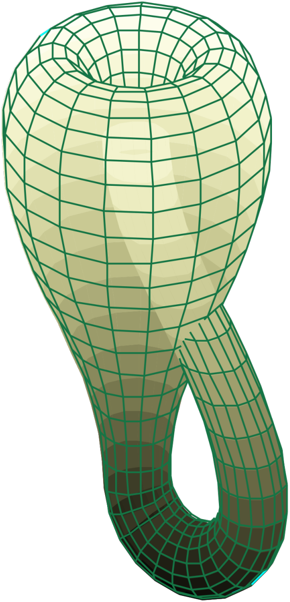
  Clearly, each equivalence class corresponds to a unique line through the origin.
- $\mathbb{RP}^2$ can also be obtained by identifying antipodal points on the sphere:

  $$\mathbb{RP}^2 = S^2 / \sim, \quad \text{where } p \sim -p.$$

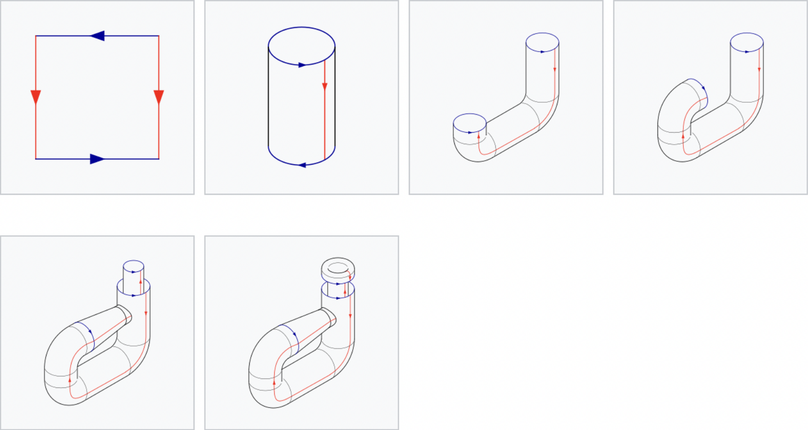
Projective Plane



Klein bottle



Klein bottle

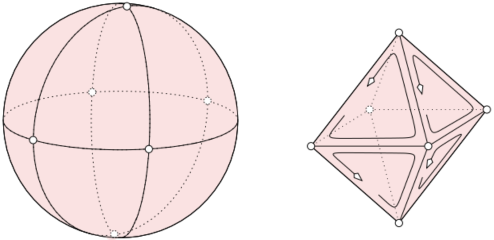


Classification of Compact 2-Manifolds

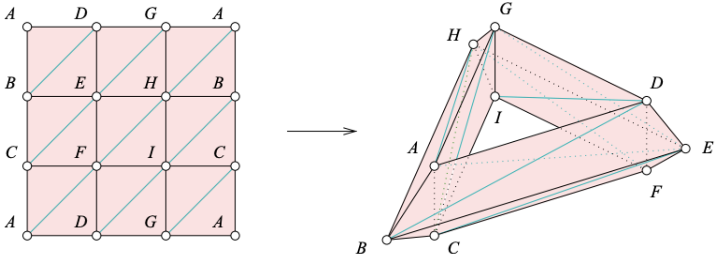
- Any compact 2-manifold is homeomorphic to:
  - A sphere S^2,
  - A connected sum of g tori, or
  - A connected sum of k projective planes.
- Orientable case: Genus g determines the surface.

Triangulation

- To **triangulate** a 2-manifold we decompose it into triangular regions.
- Each region is a disk whose boundary circle is cut at three points into three paths.
- Edges are shared by at most two triangles.
- Orientations of triangles must align along shared edges.



Triangulation of the torus



Euler Characteristic

- Defined as:

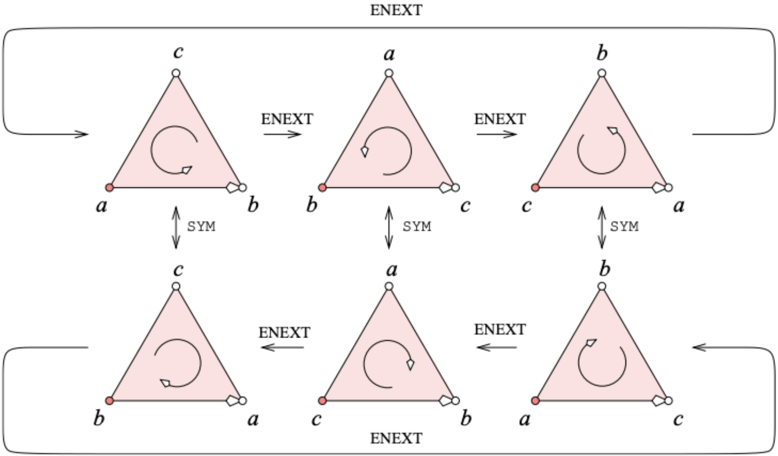
$$\chi = V - E + F,$$

where V, E, F are the number of vertices, edges, and faces in a triangulation.

- Examples:

$$\chi(S^2) = 2, \quad \chi(T^2) = 0, \quad \chi(\mathbb{RP}^2) = 1.$$

Ordered triangles



Enext and Sym Operations on Oriented Triangles

Consider an oriented triangle $T = (a, b, c)$.

- The **enext** operation cyclically permutes the vertices:

$$\text{enext}(T) = (b, c, a).$$

- The **sym** operation reverses the orientation:

$$\text{sym}(T) = (a, c, b).$$

Properties:

- Applying enext three times returns to the original triangle:

$$\text{enext}^3(T) = T.$$

- Applying sym twice returns to the original triangle:

$$\text{sym}^2(T) = T.$$

- The operations commute in the sense that:

$$\text{enext} \circ \text{sym} = \text{sym} \circ \text{enext}^{-1}.$$

Pseudocode

- A reference to the triangle consists of a pointer to a node, `m`,
- together with a integer, $\mathtt{i} \in \{1, \ldots, 6\}$, identifying the ordered version of the triangle:

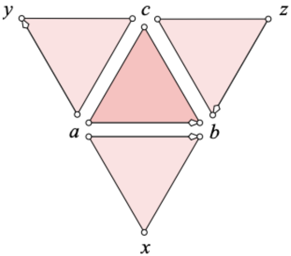
```
function enext(m,i):
    if i <= 2 then
        return (m,(i + 1) mod 3)
    else
        return (m,(i + 1) mod 3 + 4)
    endif


function sym(m,i):
    return  (m,(i + 4) mod 8)
```

Data structure to store triangulated 2-manifolds

- vertices: a linear array `V[1..n]`
- triangles: nodes of a graph
- Every triangle has exactly three neighbors $\rightarrow$ the degree of every node is three.
- Inside a node, we store pointers to the three neighbors as well as to the three vertices, which are indices into `V`

Example



- *abc* be a triangle
- x, y, z the respective third vertices of the neighbor triangles

Example

Each ordered version of the triangle points to its lead vertex and the ordered neighbor triangle that shares the directed lead edge.

In the example, assume the nodes μ, μ_x, μ_y, μ_z store the four triangles. $I = 0$ corresponds to the ordered versions *abc*, *abx*, *ayc*, *zbc* as drawn above.

Assuming *a* is stored at positions *i* in *V* and observing that *ab* is the lead edge of *abx*, the ordered triangle *abc* stores pointers $(\mu, 0).org = i$ and $(\mu, 0).fnext = (\mu_x, 0)$.

To move around in the triangulation, we use simple functions.

```
ordTri fnext(μ, I)
    return (μ, I).fnext.

int org(μ, I)
    return (μ, I).org.
```

Advanced functions

Then we can perform more advanced processes.

Depth-First-Search:

```
void Visit(μ)
  if μ is unmarked then mark μ; P1;
    forall neighbors  of μ do
      Visit()
    endfor; P2
  else P3
endif.
```

where:

P1. steps to be executed the first time the node is visited;

P2. steps to be executed after all children have been processed;

P3. steps to be executed each time the node is revisited.